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This Journal is dedicated to the following aims:

1. Through published standard papers on the culture aspects, humanism and history of mathematics to deepen and to widen public interest in its values.
2. To supply an additional medium for the publication of expository mathematical articles.
3. To promote more scientific methods of teaching mathematics.
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Vitalizing Mathematics

"Let it be distinctly understood that no question is raised here as to the vitality of mathematics. A science which has throbbed with life for untold centuries, which has imparted vigor to other sciences and furnished them indispensable aid in their growth and development, being the very backbone of engineering and the ladder on which the astronomer has ascended to the skies, and which today is expanding with a vital energy that seems to acknowledge neither bounds nor limits,—surely such a subject, big with life and all aquiver with the pulse-beat of living truth, stands in no need of being 'vitalized'.

"But the 'vitalizing' of the STUDY of mathematics is quite another and a wholly different proposition. It is assumed in this brief discussion that to 'vitalize' the study of mathematics in the high school means to use it not primarily nor chiefly for information, or application, or pleasure, but largely for the sake of the intellectual life resulting from careful mental training, from accurate thinking and clear-cut expression. In this way alone is the pupil brought 'in touch with life' to employ a popular though much abused phrase—not indeed with the life of the vegetable kingdom, or of the animal world, directly or primarily, but with soul life and the unfolding powers of the human mind."

ALFRED HUME,
University of Mississippi.

The Higher Singularities of Algebraic Curves

By B. M. WALKER
Starkville, Miss.

Introduction. The theory of the higher singularities* of algebraic curves is classed among the celebrated problems in the history of mathematics. It has invited the careful attention of mathematicians and challenged the powers of mathematical analysis for a long period of time.

The enrichment of literature, in giving to the world a clear, concise elucidation of this important theory, is indeed well worthy of the best efforts of great mathematicians, who were interested in making valuable contributions to the fund of scientific knowledge. The desire to accomplish this end and purpose stimulated keen interest and activity in the minds of mathematicians domiciled in widely situated sections of different countries.

I deem it wise to give in this paper:

1. *Brief Historical Sketch.* Some of the best mathematical achievements, mentioned in chronological order, which have been accomplished and now serve as mile-stones marking the path of progress in the development of the important theory of the higher singularities of algebraic curves.

2. *Two Fundamental Theorems.*

I. The *Noether Theorem*:† Every irreducible algebraic curve can be transformed by a birational transformation *of the plane* (a so-called *Cremona* transformation) into a curve which has *no other singular points than ordinary multiple points.*

II. The *Halphen Theorem*:‡ Every irreducible algebraic curve can be transformed by a birational transformation *of the curve* (a so-

*By *higher singularities*, we mean, in the sequel, either ordinary multiple points or points of contacts with multiple tangents. By an *ordinary multiple* point, we mean a multiple point all of whose cycles are linear and have distinct tangents. For the definition of cycles, see Halphen, "*Traité de Géométrie de Salmon,*" p. 541, and Jordan "*Cours D'Analyse*", t. 1, p. 563.

†Noether, Ueber die singulären Werthsysteme einer algebraischen Function und die singulären Punkte einer algebraischen Curve, *Mathematische Annalen*, Bd. 9, p. 166.

‡Halphen, Etude sur les points singuliers des courbes algébriques, *Traité de Géométrie de Salmon*, pp. 630, 631, 632.

called *Riemann* transformation) into a curve which has *no other singular points than ordinary double points.*

3. *On the Resolution of Higher Singularities of Algebraic Curves into Ordinary Nodes.**

A new proof of the Halphen Theorem with a geometric background in ordinary spaces of two and three dimensions.

1. *Brief Historical Sketch.*

In the scientific literature of the seventeenth century, we find that Sir Isaac Newton,† author of *Principia, Geometria Analytica, Arithmetica Universalis,* and discoverer of the Law of Gravitation, possibly the most distinguished mathematician of modern times, has given a process for determining the approximate form of an algebraic curve near the origin, either an ordinary point or a multiple point, and a good method of finding the expansions of the branches of the curve passing through the point. In his writings, he calls the curve obtained by the transformation, $y = xy$, a *hyperbolism* of the original curve.

It is highly probable that the relative importance, in the domain of mathematics, of the nature and analysis of the higher singularities of algebraic curves dates back to the time of Euler‡ and Cramer.§ It appears that Euler was the first mathematician to observe the paradox in mathematics, that two algebraic curves of the nth degree may intersect in a greater number of points than are necessary to determine a curve of the nth degree. Cramer made the same observation. In making successive transformations, he used the form, $y = vx$, and showed certain singularities with coinciding tangents occur as final forms of singularities with distinct tangents involving a number of ultimately vanishing loops.

To Plücker,‡ the credit is due for having made the first systematic study of the theory of the singularities of algebraic curves. His contribution to the fund of knowledge in this domain of mathematics is indeed most highly note-worthy. Plücker's well known "character-

*Dissertation, Chicago, (1906).

†Newton, De reductione affectarum equationum (Opusc. ed. Castillon, Vol. 1, p. 37); Analysis per equationes numero terminorum infinitas, (1669); Enumeratio linearum tertii ordinis, (1706). See also Puiseux, *Liouville,* t. 15, (1850) and t. 16, (1851); Stolz, *Mathematische Annalen,* Bd. 8, p. 433; Hamburger, *Zeitschrift für Math. und Physik.* Bd. 16.

‡Euler, On an apparent Contradiction in the Theory of Curves, *Berlin Transaction,* (1748).

§Cramer, Introduction a l'Analyse des lignes courbes algébriques, (1750).

‡Plücker, Solution d'une question fondamental concernant la théorie generale des Courbes, *Crelle,* Bd. 12, p. 105; Theorie der algebraischen Curven. See also Cayley, *Crelle,* Bd. 34, p. 30.

istics" and equations relative to ordinary singularities of algebraic curves, which are capable of extension to include higher singularities, are considered as standard authority throughout the world and will serve all higher students of mathematics most usefully during the coming centuries of time.

Since the days of Plücker, various phases of the problem have been studied extensively by mathematicians domiciled in England, the different countries of Europe, America and other parts of the world. For the more important pieces* of literature on the subjects, reference is duly made. This literature includes many problems connected more or less closely with the theory of the singularities of algebraic curves and will be interesting and highly instructive reading to any student of higher mathematics.

2. *Two Fundamental Theorems.*

Noether's Theorem: Every irreducible algebraic curve can be transformed by a birational transformation *of the plane* (a so-called *Cremona* transformation) into a curve which has *no other singular points than ordinary multiple points.*

This fundamental theorem of great importance in the theory of higher singularities of algebraic curves stands like a mountain peak upon which mathematicians have rested analytically and were able to plan most carefully their different methods of approach to the solution of the celebrated Halphen Theorem.†

*Bertini, Trasformazione di una curva algebrica in un'altra con soli punti doppi, *Math. Ann.*, Bd. 44, p. 158.

Brill, Ueber Singularitäten ebener algebraischen Curven, *Math. Ann.*, Bd. 16, p. 348.

Cayley, On the higher singularities of a plane curve, *Quarterly Journal*, Vol. 7, p. 212, and *Crelle*, Bd. 64, p. 369.

De La Gournerie, Notes sur les singularités éleveés des courbes planes, *Journal de Mathématiques*, 2 series, t. 14, p. 425.

Halphen, Sur les points singuliers des courbes algébriques planes, Memoires presentes par divers savants a l'Académie des Sciences, t. 26; *Comptes Rendus*, t. 78; also *Traité de Géométrie de Salmon.*

Noether, Ueber die singulären Werthsysteme einer algebraischen Function und die singulären Punkte einer algebraischen Curve, *Math. Ann.*, Bd. 9, p. 166. Compare also, Picard, Traité D'Analyse, t. 11, p. 364, and Jordan, Cours D'Analyse, t. 1, p. 588.

Scott, On the Higher Singularities of Plane Curves, *American Journal of Mathematics*, Vol. 14, pp. 301-325.

Simart, Sur un théoréme relatif a la transformation des courbes algébriques, *Comptes Rendus*, t. 116.

Smith, On the Higher Singularities of Plane Curves, *Proceedings of the London Mathematical Society*, Vol. 6, p. 153.

Stolz, Ueber die singulären Punkte der algebraischen Functionen und Curven, *Math. Ann.*, Bd. 8, p. 415.

Zeuthen, Note sur les singularités des courbes planes, *Mathematische Annalen*, Bd. 10, p. 210.

†Halphen, Traité de Géométrie de Salmon, pp. 630, 631, 632.

In 1871, Noether* first announced this theorem ascribed to him and gave a method of solution.

In 1875 and 1876,† Halphen suggested a method of treatment to transform an algebraic curve with higher singularities into another curve possessing only ordinary multiple points by means of birational transformations.

In his method of proof, he would place the origin of coordinates to coincide with one of the singular points to be resolved and then make use of a quadratic transformation. The transformed curve would show the singular points separated from the origin to have remained unchanged in nature and orders of multiplicity and with the singular point at the origin to have disappeared; but, corresponding to it, other singular points to have made appearance at different places on the curve; thus, leaving in doubt whether anything had been accomplished by the transformation.

Noether‡ in 1876 gave a revised and more elaborate discussion of the problem, in an elegant paper, treating the analysis of the singular points of algebraic curves, which successfully established this theorem of such great importance. However, since that time, the theorem has been studied very carefully by Bertini,§ by Picard,‡ who gave a proof ascribed by him to Simart, and by many others.¶

Halphen's Theorem:|| Every irreducible algebraic curve can be transformed by a birational transformation *of the curve* (a so-called *Riemann* transformation) into another curve which has *no other singular points than ordinary double points.*

It seems that this fundamental theorem was first enunciated by Halphen in 1884. His proof, however, was incomplete as has been pointed out by Picard.** Though it appears from the literature that knowledge of the theorem was in the minds of mathematicians prior to that time. Halphen himself ascribed the theorem to Noether and

*Noether, *Goettinger Nachrichten* (1871); *Math. Ann.*, Bd. 9.

†Halphen, *Comptes rendus*, t. 80; Oeuvres, T. 1; and *Jour. de Math.* t. 2.

‡Noether, Loc. cit.

§Bertini, *Lombardo Rendiconti*, Vol. 21, p. 326.

‡Picard, *Traité D'Analyse*, t. 2, pp. 360-364. (1893).

¶Brill und Noether, Die Entwickelung der Theorie der algebraischen Functionen, *Jahresbericht der Vereinigung*, Bd. 3, p. 369.

Clebsch-Lindemann, Vorlesungen über Geometrie, Bd. 1, p. 491.

Encyclopädie, 111 C. 4, p. 363.

Enriques, Lezioni sulla Teoria Geometrica delle Equasioni e delle Funzioni Algebriche, Vol. 2, p. 417.

Jordan, Cours D'Analyse, t. 1, 2nd ed. p. 588, (1893).

Pascal, Repertorium der Höheren Mathematik, Bd. 2, 2nd ed. p. 291.

Severi, Lezioni di Geometria Algebrica, p. 61.

||Halphen, Loc. cit.

**Picard, Traité D'Analyse, t. 2, p. 364, (1893).

Clebsch had oral communication with Klein concerning it as early as 1869.

Klein* in his *Riemannche Flächen* makes reference to the subject and asks the question, "Wo ist der Satz zum ersten Male gedruckt?" Kronecker† in 1881 published a paper treating the analysis of the discriminant of an algebraic function, which he claims he had presented to the Berlin Academy as early as 1862; and his reasoning and results are highly interesting in connection with this theorem.

Hensel and Landsberg‡ are inclined to ascribe the origin of it to Kronecker and in 1902 greatly extended Kronecker's line of reasoning. But it appears to me that Halphen's enunciation was the first published announcement making an explicit statement of this problem with a suggested method of proof and that a prior satisfactory solution of it has not been made. The Halphen announcement coupled with Picard's criticism seemed to be the electric spark that touched off the mathematical world and greatly stimulated the efforts and activities of mathematicians to bring forth a correct and satisfactory solution of this celebrated problem.

In addition to the Kronecker line of reasoning, three§ essentially different methods have been employed by groups of mathematicians for the final proof:

1). Picard‡ uses a quadratic one-to-three transformation of the plane. In 1893, he reproduced Halphen's argument. In the same year, Simart¶ gave a modified form of Halphen's line of reasoning. The problem was likewise studied by Appell and Goursat|| in 1895, and by Vessiot** in 1896. The lines of reasoning seem plausible but in places the steps are analytically delicate and with processes of my own analysis I am in doubt if the final solution is possible by the method of a quadratic one-to-three transformation of the plane.

2). Bertini†† uses, for the same purpose, a cubic one-to-two transformation of the plane. In 1891, he published his paper in the *Rivista di Matematica* and again in 1894, in the *Mathematische Annalen*

*Klein, Riemannche Flächen, p. 245, (1892).

†Kronecker, *Journal Für Mathematik*, Bd. 91, p. 301, (1881).

‡Hensel and Landsberg, Theorie der algebraischen Functionen einer Variabeln pp. 402-409.

§Berzolari, Encyclopädie, 111 C 4, p. 362; see also Brill and Noether, *Jahresbericht der Vereinigung*, Bd. 3; Pascal, Repertorium der Höheren Mathematik, Bd. 2, 2nd ed.; Wirtinger, Encyclopädie 11, B 2, p. 127.

‡Picard, Traité D'Analyse, t. 11, p. 366, (1893).

¶Simart, *Comptes rendus de l'Académie*, t. 11, p. 366, (1893).

||Appell and Goursat, Theorie des Fonctions Algébrique, *Journal de Mathematiques*, p. 282.

**Vessiot, *Annales de Toulouse*, t. 10, (1896).

††Bertini, *Rivista di Matematica*, t. 1, p. 32; *Mathematische Annalen*, Bd. 44, p 158; see also Clebsch and Lindemann, Vorlesungen über Geometrie, p. 661, (1876).

with a foot-note addition. To this last paper, Klein added a second foot-note of great importance, which he says was communicated to him orally by Clebsch in 1869. In 1905, Picard* adopted Bertini's proof with only some minor modifications and published it in his *Traité d'Analyse.*

3). Poincaré† transforms the plane curve into a twisted curve in space of higher dimensions which has no singular points and then shows that this curve may be projected into a plane curve with no other singular points than ordinary double points.

Using in the main procedure this same line of geometric reasoning, other mathematicians‡ have made their contributions to the literature on the subject.

Bliss§ in a Presidential address before the American Mathematical Society reviews the groups of proofs which he classifies as the Kronecker Group, the Halphen Group, The Bertini Group, and Other Geometric Proofs. All of these proofs prior to 1906, however, are written in an exceedingly concise style and leave a great many minor points to the reader.

We shall consider briefly the Bertini proof:

> *Trasformazione di una curva algebrica in un'altra con soli punti doppi.*

Let $f=0$ be any irreducible algebraic plane curve which has no other singular points than a finite number of ordinary multiple points $A_0, A_1, A_2, \ldots, A_r$, with orders of multiplicity greater than 2.

By the use of a cubic one-to-two transformation of the plane, he proposes to transform the curve, $f=0$, into another curve, $f'=0$, which has no other singular points than ordinary double points. The transformation system consists of a net of cubics through seven fundamental points, $B_0, B_1, \ldots, B_6$, with the fundamental point B_0 so chosen as to coincide with the multiple point A_0, of order n, on the curve, $f=0$, to be resolved; with the fundamental points B_1, B_2, B_3 B_4, B_5, taken arbitrarily in the plane, but not to lie on the curve, $f=0$; and with the fundamental point B_6, so chosen that it shall not fall on certain loci, $\lambda_1, \lambda_2, \lambda_3, \lambda_4$, in the plane. Locus λ_1 is used to avoid

*Picard, Traité d'Analyse, t. 2, 2nd ed. p. 408, (1905).

†Poincaré, *Comptes rendus de l'Académie*, t. 117, p. 18, (1893).

‡Del Re, *Modena Memorie*, 10, p. 447, (1894); Del Pezzo, *Napoli Rendiconti*, t. 7, p. 15, (1893); Encyclopädie der Mathematischen Wissenschaften, Bd. 3, (1906); Jamet, *Nouvelles Annales*, t. 19, p. 500, (1900); Pieri, *Rivista di Matematica*, t. 4, p. 40, (1894); Segre, *Annali di Matematica*, t. 25, p. 43, (1897); Severi, Lezioni di Geometria Algebrica, p. 172, (1908); Veronese, *Mathematische Annalen*, Bd. 19, p. 213, (1881); Vessiot, *Bull. de la Société Math. de France*, t. 22, (1894).

§Bliss, *Bull. Am. Math. Soc.*, Vol. 29, p. 161; *Trans. Amer. Math. Soc.*, Vol. 24, (1922).

a one-to-two correspondence between the transformed curve, $f_1=0$, and the curve, $f=0$.

To determine this locus: Pass a pencil of cubics through the points $A_0, B_1, \ldots, B_5$, and two points on the curve, $f=0$; one point P_1 being fixed, chosen arbitrarily, and the other point P being a variable point. The ninth base point Q_1 of the pencil of cubics describes the locus λ_1 as the variable point P describes the curve, $f=0$.

Locus λ_2 is used to prevent linear cycles with distinct centers from being transformed into cycles of higher order.

To determine this locus: Pass a pencil of cubics through the six points $A_0, B_1, \ldots, B_5$, and to touch the curve $f=0$, in a variable point P. The ninth base point Q_2 of the pencil will describe the locus λ_2 as the variable point P describes the curve, $f=0$.

Locus λ_3 is used to prevent linear cycles with coinciding centers from being transformed into cycles of higher order.

To determine this locus: Pass a pencil of cubics through the six points $A_0, B_1, \ldots, B_5$ and to touch an arbitrary line at the point A_i, $i=1,2,\ldots, r$; the ninth base point Q_{3i} of the pencil of cubics will describe a curve of the locus as the arbitrary line revolves about the point A_i. The totality of these r curves makes up the locus λ_3.

Locus λ_4 is used to prevent linear cycles with some coinciding centers, but not all, from being transformed into new cycles with all centers coinciding in the same point.

To determine this locus: Pass a pencil of cubics through the six points $A_0, B_1, \ldots, B_5$, a point A_i, $i=1,2,\ldots, r$, and a variable point P on the curve, $f=0$; the ninth base point Q_{4i} of the pencil of cubics will describe a curve of the locus as the variable point P describes the curve, $f=0$. The totality of these r curves makes up the locus.

In my opinion, three additional loci, $\lambda_5, \lambda_6, \lambda_7$, will strengthen the validity of the solution.

Locus λ_5 is necessary for use to prevent linear cycles with centers coinciding in the multiple point A_0 from being transformed into new cycles of higher order.

To determine this locus: Pass a pencil of cubics through the six points $A_0, B_1, \ldots, B_5$, to touch an arbitrary line at the point A_0, and with the additional condition imposed, that, in the multiple point A_0, of order n, the pencil of cubics shall have three points in common with the linear cycle C_i, $i=1,2,\ldots, n$; the pencil is attached to the cycle.

As the arbitrary line revolves about the point A_0, the ninth base point, Q_{5i}, of the pencil describes a curve of the locus. The totality of these n curves makes up the locus.

Locus λ_6 is necessary for use to prevent coinciding tangents in the new double points arising out of distinct ordinary points on the curve, $f=0$.

To determine this locus: Pass a pencil of cubics through the points $A_0, B_1, \ldots, B_5$, a variable point P, and a point P_i, $i=1,2,\ldots,6$, on the curve, $f=0$. The point P_i is determined by passing a cubic φ_i through the points $A_0, B_1, \ldots, B_5$, and P, under the condition to be tangent to the curve, $f=0$, at the point P and to be tangent to the curve, $f=0$, at some other point. The cubic φ_i is completely determined and this point of tangency with the curve, $f=0$, is the required point P_i, $i=1,2,\ldots,6$. The ninth base point Q_{6i} of the pencil of cubics will describe a curve of the locus as the variable point P describes the curve, $f=0$. The totality of the 6 curves makes up the locus.

Locus λ_7 is necessary for use to prevent coinciding tangents in new double points arising out of the multiple point A_0.

To determine this locus: Pass a pencil of cubics through the points $A_0, B_1, \ldots, B_5$, meeting the linear cycle C_i, $i=1,2,\ldots,n$, in two points at the point A_0, and passing through an additional point P_i, on the curve, $f=0$. The point P_i is determined by passing a cubic ψ_i through the points $A_0, B_1, \ldots, B_5$, to meet the linear cycle C_i, in three points at the point A_0, and to touch the curve $f=0$ at some other point. The cubic ψ_i is completely determined and the point of tangency with the curve, $f=0$, is the required point P_i. As the point P_i describes the curve, $f=0$, the ninth base point Q_{7i} of the pencil of cubics will describe a curve of the locus. The totality of the n curves makes up the locus.

3. *On the Resolution of Higher Singularities of Algebraic Curves into Ordinary Nodes.*

The object of this paper is to give a new proof of the *Halphen* theorem, in question, by carrying out, in detail, an idea contained in the following foot-note added by Professor *Felix Klein* to Bertini's paper:

"Die Methode von Bertini kommt geometrich zu reden darauf zuruck, die Ebene, in welcher uns die Curve mit singulären Punkte gegeben ist, als eindeutigen Abbildung einer Fläche 3 Ordnung zu betrachten, dadurch die Curve in eine Raumcurve zu verwandeln und letztere hinterher wieder von einen hinreichend allgemeinen Punkte aus auf eine andere Ebene zu projiciren. In dieser Form ist mir der Ansatz noch von *Clebsch* her bekannt, der mir denselben in Herbst 1869 mündlich mittheilte".

1. *Triply infinite linear systems of cubics.*

We shall develop first a number of auxiliary theorems on triply infinite linear systems of cubics which will be needed in our discussion of the one-to-one correspondence between the plane and the cubic surface.

In a plane Π, whose points are referred to a system of trilinear coordinates $x:y:z$, there are given six points $P_i(x_i, y_i, z_i)$, $(i=1,2\ldots,6)$ chosen so that *no three points lie on a straight line.*

We consider the problem: To determine all cubics passing through the six points.

Let

$$(1)\qquad f\equiv Ax^3+By^3+Cz^3+Dx^2y+Ex^2z \\ +Fxy^2+Gxz^2+Hy^2z+Iyz^2+Jxyz=0,$$

where $A, B, C, \ldots,$ are arbitrary constants and $x, y, z,$ are homogeneous coördinates, be the general equation of a cubic. If the cubic, $f=0$, is to pass through the six points P_i, the coefficients $A, B, C, \ldots,$ must satisfy the six equations

$$(2)\qquad Ax^3_i+By^3_i+Cz^3_i, \ldots =0, \quad (i=1,2,\ldots, 6).$$

The matrix of the coefficients,

$$M\equiv\left\|x_i^3\ y_i^3\ z_i^3\ x_i^2y_i\ x_i^2z_i\ x_iy_i^2\ x_iz_i^2\ y_i^2z_i\ y_iz_i^2\ x_iy_iz_i\right\|,$$

is always of rank six; that is, at least, one of its determinants of order six is $\neq 0$. If we choose the vertices of the triangle of reference to coincide with the points P_1, P_2, P_3, respectively, it can be shown that the determinant

$$\begin{vmatrix} x^2_4y_4 & x_4y^2_4 & x_4y_4z_4 \\ x^2_5y_5 & x_5y^2_5 & x_5y_5z_5 \\ x_6^2y_6 & x_6y_6^2 & x_6y_6z_6 \end{vmatrix}$$

to which, in the present case, one of the determinants of M reduces, is $\neq 0$, and consequently the rank of M, which is invariant under a transformation of coordinates, is *six*. At the same time, we have for this special triangle of reference, $A=B=C=0$, in (2) and easily obtain the totality of cubics through the six points in the form (3)

$$(3)\qquad \sum\equiv\alpha f_1+\beta f_2+\sigma f_3+\delta f_4=0,$$

$\alpha, \beta, \sigma, \delta,$ being arbitrary constants; and then finally show that the four cubics $f_1, f_2, f_3, f_4,$ are linearly independent. Hence *the cubics through the six points form, without exception, a triply infinite system, provided that no three of the six points lie on a straight line.*

§2. *Reduction of the base cubics to canonical form.*

Instead of the four base cubics f_1, f_2, f_3, f_4, we introduce four other base cubics each of which breaks up into three straight lines. Calling the new base cubics f_1, f_2, f_3, f_4, again, it is easily shown that their expressions are as follows:

$$f_1 \equiv P_4P_6 \cdot P_1P_3 \cdot P_2P_5,$$

$$f_2 \equiv P_4P_6 \cdot P_2P_3 \cdot P_1P_5,$$

$$f_3 \equiv P_4P_5 \cdot P_1P_3 \cdot P_2P_6,$$

$$f_4 \equiv P_4P_5 \cdot P_2P_3 \cdot P_1P_6,$$

and the following results are easily demonstrated:

1) The base cubics have no other common points of intersection except the six points P_i, $(i=1,2,\ldots, 6)$.
2) Not one of the points, P_i, is a multiple point on all the base cubics.
3) At none of the points, P_i, have all the base cubics coinciding tangents.

§3. *Rank of the matrix of the first derivatives of*

$$f_1, f_2, f_3, f_4.$$

We consider the matrix

$$\left\|\begin{matrix} f_{1x} & f_{2x} & f_{3x} & f_{4x} \\ f_{1y} & f_{2y} & f_{3y} & f_{4y} \\ f_{1z} & f_{2z} & f_{3z} & f_{4z} \end{matrix}\right\|$$

where $$f_{i_x} \equiv \frac{\partial f_i}{\partial x}, \qquad f_{i_y} \equiv \frac{\partial f_i}{\partial y} \qquad f_{i_z} \equiv \frac{\partial f_i}{\partial z}$$

and propose to determine all points in the plane for which the rank r of the matrix is 0, 1, 2, respectively.

Taking into consideration the determinants of the matrix, we find there are no points in the plane in which the rank r of the matrix is $r=0$, or $r=1$. When $r=2$, all determinants of order *three* vanish; whereas, at least, one determinant of order *two* is $\neq 0$.

Such points are the common points of intersection of the Jacobians of the four nets of cubics:

$$(4) \quad \begin{cases} \alpha f_1+\beta f_2+\sigma f_3+\delta f_4=0, & \alpha' f_1+\beta' f_2+\sigma' f_3+\delta' f_4=0, \\ \alpha'' f_1+\beta'' f_2+\sigma'' f_3+\delta'' f_4=0, & \alpha''' f_1+\beta''' f_2+\sigma''' f_3+\delta''' f_4=0. \end{cases}$$

After considering the Jacobians of the four nets of cubics, we find: *There are six, and only six, points in the plane* Π, *in which* $r=2$; *namely, the six fundamental points*; *and* $r=3$, *at all other points.*

§4. *Transformation from plane to cubic surface.**

As before, let $P_1, \ldots, P_6$ be six distinct points in a plane Π_1 no three of which lie on a straight line, and further let

$$(5)\qquad \begin{cases} f_1(\lambda,\mu,\nu)=0, & f_2(\lambda,\mu,\nu)=0, \\ f_3(\lambda,\mu,\nu)=0, & f_4(\lambda,\mu,\nu)=0, \end{cases}$$

where λ, μ, ν, denote any system of trilinear coordinates, be the four base cubics of the triply infinite system passing through the six points.

Then, if $x_1 : x_2 : x_3 : x_4$ denote tetrahedral coordinates of a point in space, the equations

$$(6)\qquad \rho x_1=f_1,\ \rho x_2=f_2, \rho x_3=f_3, \rho x_4=f_4,$$

define, in parameter representation, a cubic surface F_3 and, at the same time, a correspondence between the points of the plane and the points of the cubic surface.

We find:

a) To any point, P_0, in the plane Π, different from the points $P_1, \ldots, P_6$ corresponds one point P'_0 on the surface F_3; since, not all four of the quantities, $f_1{}^\circ, f_2{}^\circ, f_3{}^\circ, f_4{}^\circ$, equal zero, and the ratios $x_1 : x_2 : x_3 : x_4 = f_1{}^\circ, : f_2{}^\circ : f_3{}^\circ : f_4{}^\circ$, have definite values; $f_1{}^\circ, f_2{}^\circ, f_3{}^\circ, f_4{}^\circ$ are the values of the functions f_1, f_2, f_3, f_4, respectively, at the point P_0. The six points $P_1, \ldots, P_6$, for which the functions, f_1, f_2, f_3, f_4, vanish simultaneously are called *fundamental points.*

We find further:

b) To a fundamental point, together with the totality of the paths of approach to it, corresponds a fundamental curve on F_3.

c) Two different lines of approach to P_0 determine two different points on the fundamental curve.

d) A fundamental curve on F_3 is a straight line. There are, therefore, six fundamental lines on F_3 corresponding to the six fundamental points in the plane.

*Clebsch, Geometrie aut den Flächen dritter Ordnung, *Crelle*, 65.

§5. *The one-to-one character of the correspondence between the plane and the cubic surface.*

Considering this correspondence, we find:

a) To two non-fundamental points in the plane always correspond two different points on the cubic surface, *provided the additional condition be imposed upon the fundamental points, that the six points, P_1, P_2, P_3, P_4, P_5, P_6, do not lie on a conic.*

b) The image of a non-fundamental point does not lie on a fundamental line.

c) The image of a non-fundamental point is not a singular point on the cubic surface.

d) Two fundamental lines on the surface F_3 have no common point.

As a result of these theorems, it follows, *that to every point of F_3, not on a fundamental line, corresponds one, and but one, point in* Π, *without exception*; *and to every point of F_3, on a fundamental line, corresponds, in* Π, *one, and but one, fundamental point together with one, and but one, path of approach to it, without exception*; *there are, therefore, no fundamental curves in the plane* Π.

Furthermore, we find *there are twenty-seven, and only twenty-seven, straight lines on the surface F_3.*

§6. *Generalities about cycles.*

Before considering the transformation of an algebraic curve from the plane upon the cubic surface, we find it necessary to give some consideration to generalities about cycles.

Let

$$K: \qquad G(\lambda,\mu,\nu)=0 \tag{1}$$

be an algebraic curve in the plane Π, then according to the theory of the singularities of algebraic curves* all points of K in the vicinity of a given point $P_0(\lambda_0,\mu_0,\nu_0)$ of K are expressible by means of a finite number of convergent power series of an auxiliary variable t (the parameter) as follows:

$$\begin{aligned} \rho\lambda &= \lambda_0+\lambda_1 t+\lambda_2 t^2+\ldots, \\ \rho\mu &= \mu_0+\mu_1 t+\mu_2 t^2+\ldots, \\ \rho\nu &= \nu_0+\nu_1 t+\nu_2 t^2+\ldots, \end{aligned} \qquad (C) \tag{2}$$

called according to *C. Jordan*† "cycles".

*Puiseux, Recherches sur les fonctions algébriques; Liouville, *Journal de Mathématiques*, t. 15, p. 365; Weierstrass, Theorie der Abelchen Transcendenten, Bd. 4, p. 13; Hamburger, Ueber die Entwickelung algebraischen Functionen in Reihen, *Zeitschrift für Mathematik und Physik*, Bd. 16.

†Jordan, Cours D'Analyse, t. 1, p. 561.

Jordan shows (Cours D'Analyse t. 1, p. 563) that the parameters of the points of intersection of a straight line

$$(3) \qquad A\lambda + B\mu + C\nu = 0,$$

with the cycle (C) are given by the equation

$$(4) \qquad A\lambda_0 + B\mu_0 + C\nu_0 + (A\lambda_1 + B\mu_1 + C\nu_1)t + \ldots = 0.$$

He also gives the necessary and sufficient conditions to define the order of the cycle and the tangent to the cycle. These definitions concerning cycles can be extended to curves in space of three dimensions. Let

$$(5) \qquad \begin{cases} \rho x_1 = K_0 + K_1 t + K_2 t^2 + \ldots, \\ \rho x_2 = L_0 + L_1 t + L_2 t^2 + \ldots, \\ \rho x_3 = M_0 + M_1 t + M_2 t^2 + \ldots, \\ \rho x_4 = N_0 + N_1 t + N_2 t^2 + \ldots, \end{cases} \qquad (C')$$

be a cycle in space, where K_0, L_0, M_0, N_0, the coördinates of P_0', the center of the cycle, are not all equal to zero. Then the parameters of the points of intersection of an arbitrary plane

$$(6) \qquad Ax_1 + Bx_2 + Cx_3 + Dx_4 = 0,$$

with the cycle (C') are given by the equation

$$(7) \quad AK_0 + BL_0 + CM_0 + DN_0 + (AK_1 + BL_1 + CM_1 + DN_1)t + \ldots = 0.$$

It is easy to define the order of the cycle and show the tangent to the cycle in parameter representation.

§7. *Transformation of the non-singular points of the algebraic curve to the cubic surface.*

We now apply the results of the preceding sections to the resolution of the singular points of an algebraic curve into double points. Let

$$(8) \qquad K: \qquad G(\lambda, \mu, \nu) = 0,$$

be the algebraic curve in the plane Π; whose singularities we wish to resolve. We suppose that the curve K is irreducible and that it has r *ordinary multiple points*, $A_1, \ldots, A_r$, with only linear cycles and with distinct tangents, and no other singular points.* We may further suppose without loss of generality that the degree of the curve K is greater than *three*.

*Noether, *Mathematische Annalen*, Bd. 9, p. 166.

We let one of the six fundamental points coincide with the multiple point A_1 to be resolved, but choose all other fundamental points external to K. The image of K is a curve K' on F_3; we propose to study the properties of this curve K'.

Let P_0' be any point of the curve K'; according to §5, P_0' is the image of one, and but one, point P_0 of K. The point P_0 is either an ordinary point, or one of the multiple points, $A_2, \ldots, A_r$, or the multiple point A_i. If P_0 is an ordinary point, then all points of K in the vicinity of P_0 are represented by one linear cycle (C),

$$(9)\qquad \begin{cases} \rho\lambda = \lambda_0 + \lambda_1 t + \lambda_2 t^2 + \ldots, \\ \rho\mu = \mu_0 + \mu_1 t + \mu_2 t^2 + \ldots, \\ \rho\nu = \nu_0 + \nu_1 t + \nu_2 t^2 + \ldots, \end{cases} \qquad (C)$$

where not all of the quantities λ_0, μ_0, ν_0 are equal zero and

$$(10)\qquad \left\| \begin{matrix} \lambda_0 & \mu_0 & \nu_0 \\ \lambda_1 & \mu_1 & \nu_1 \end{matrix} \right\| \neq 0.$$

The image of the cycle (C) is a cycle (C'),

$$(11)\qquad \begin{cases} \rho' x_1 = K_0 + K_1 t + K_2 t^2 + \ldots, \\ \rho' x_2 = L_0 + L_1 t + L_2 t^2 + \ldots, \\ \rho' x_3 = M_0 + M_1 t + M_2 t^2 + \ldots, \\ \rho' x_4 = N_0 + N_1 t + N_2 t^2 + \ldots, \end{cases} \qquad (C')$$

a cycle of the curve K'; where

$$(12)\qquad \begin{cases} K_0 = f_1^\circ,\ K_1 = \lambda_1 f_{1\lambda}^\circ + \mu_1 f_{1\mu}^\circ + \nu_1 f_{1\nu}^\circ, \ldots; \\ L_0 = f_2^\circ,\ L_1 = \lambda_1 f_{2\lambda}^\circ + \mu_1 f_{2\mu}^\circ + \nu_1 f_{2\nu}^\circ, \ldots; \\ M_0 = f_3^\circ,\ M_1 = \lambda_1 f_{3\lambda}^\circ + \mu_1 f_{3\mu}^\circ + \nu_1 f_{3\nu}^\circ, \ldots; \\ N_0 = f_4^\circ,\ N_1 = \lambda_1 f_{4\lambda}^\circ + \mu_1 f_{4\mu}^\circ + \nu_1 f_{4\nu}^\circ, \ldots. \end{cases}$$

It is easily shown that (C') is a linear cycle of the curve K'; hence *the image P_0' of an ordinary point P_0 of K is an ordinary point of K'.*

§8. *Transformation of the multiple points, $A_2, \ldots, A_r$, to the cubic surface.*

If P_0 is one of the multiple points, $A_2, \ldots, A_r$ of order of multiplicity σ, then all points of K in the vicinity of P_0 are represented by σ linear cycles with coinciding centers and with distinct tangents. Let (9) and

$$(13)\qquad \begin{cases} \rho\lambda = \lambda_0 + \lambda'_1 t + \lambda'_2 t^2 + \ldots, \\ \rho\mu = \mu_0 + \mu'_1 t + \mu'_2 t^2 + \ldots, \\ \rho\nu = \nu_0 + \nu'_1 t + \nu'_2 t^2 + \ldots, \end{cases} \qquad (C_1)$$

be two of these cycles, then

$$(14) \qquad \left\|\begin{matrix} \lambda_0 & \mu_0 & \nu_0 \\ \lambda'_1 & \mu'_1 & \nu'_1 \end{matrix}\right\| \neq 0,$$

and since the tangents to the two cycles are distinct, we must have, besides,

$$(15) \qquad \begin{vmatrix} \lambda_0 & \mu_0 & \nu_0 \\ \lambda_1 & \mu_1 & \nu_1 \\ \lambda'_1 & \mu'_1 & \nu'_1 \end{vmatrix} \neq 0;$$

and the image (C'_1) of (C_1) is given by the equations

$$(16) \qquad \begin{cases} \rho' x_1 = K_0 + K_1' t + K_2' t^2 + \dots, \\ \rho' x_2 = L_0 + L_1' t + L_2' t^2 + \dots, \\ \rho' x_3 = M_0 + M_1' t + M_2' t^2 + \dots, \\ \rho' x_4 = N_0 + N_1' t + N_2' t^2 + \dots, \end{cases} \qquad (C_1')$$

where $\qquad K_1' = \lambda'_1 f_{1\lambda}{}^\circ + \mu_1' f_{1\mu}{}^\circ + \nu_1' f_{1\nu}{}^\circ$, etc.

It is then easily shown that all the points of K' in the vicinity of P_0' are represented by σ linear cycles with coinciding centers and with distinct tangents; that is, the image A'_i of one of the multiple points A_i $(i=2,3,\dots, r)$ of multiplicity σ_i is *an ordinary multiple point of order σ_i of K'.*

§9. *Transformation of the multiple point A_1 to the cubic surface.*

If the point P_0 is the multiple point A_1, of any order of multiplicity σ_1, then all points of K in the vicinity of P_0 are represented by σ_1 linear cycles with coinciding centers and distinct tangents.

According to our choice of fundamental points, the point P_0 or A_1 coincides in this case with one of the fundamental points. Hence, if (9) represents one of the σ_1 cycles, (C), with center P_0 and (11) its image (C') on F_3, then

$$(17) \qquad K_0=0,\ L_0=0,\ M_0=0,\ N_0=0,$$

and after division by t the cycle (C') takes the form

$$(18) \qquad \begin{cases} \rho x_1 = K_1 + K_2 t + K_3 t^2 + \dots, \\ \rho x_2 = L_1 + L_2 t + L_3 t^2 + \dots, \\ \rho x_3 = M_1 + M_2 t + M_3 t^2 + \dots, \\ \rho x_4 = N_1 + N_2 t + N_3 t^2 + \dots. \end{cases} \qquad (C')$$

Without serious difficulty it can be shown that the cycle (C') of the curve K' is a linear cycle.

According to §4, the centers of the σ_1 linear cycles of the curve K' are all distinct, distributed along the fundamental line, each center being determined by the tangent line to the corresponding cycle of the curve K; and according to §5, each center of the σ_1 linear cycles of K' does not have the center of any other linear cycle to coincide with it. Hence, *each point of the σ_1 points of K' is an ordinary point.*

Therefore *the curve K is transformed into a new curve K' on the surface F_3, given by the equations*

$$(19) \qquad \rho x_1 = f_1(\lambda,\mu,\nu),\ \rho x_2 = f_2(\lambda,\mu,\nu),\ \rho x_3 = f_3(\lambda,\mu,\nu),\ \rho x_4 = f_4(\lambda,\mu,\nu),$$

combined with the equation

$$G(\lambda,\ \mu,\ \nu) = 0,$$

which has the multiple point A_1 of K resolved into ordinary points and all the remaining multiple points, $A_2, \ldots, A_r$, of K transformed into new multiple points, $A'_2, \ldots A'_r$, unchanged in orders of multiplicity, with linear cycles and with distinct tangents, and without the introduction of any new multiple points or other singularities.

Finally, we consider the transformation from the surface F_3 to a plane Π'.

§10. *One-to-two correspondence between the surface F_3 and the plane Π'.*

We now select a point O on F_3, and a plane Π' not passing through O, and project K' from O to Π', thus obtaining a plane algebraic curve K''. We are going to prove, that the point O can be so chosen, that the correspondence thus established between the curves K' and K'' is a one-to-one correspondence of points and that the curve K'' has $r-1$ ordinary multiple points, $A''_2, \ldots, A''_r$, of the same multiplicities as $A_2, \ldots, A_r$ of K, and besides no other multiple points, except ordinary double points.

Formulae for the correspondence between F_3 and Π': We introduce a new tetrahedron of reference, one of whose vertices, $\xi=0$, $\eta=0$, $\zeta=0$, $T=1$, coincides with the point O, while the opposite face ABC, $T=0$, coincides with the plane Π'.

If ξ_0, η_0, ζ_0, T_0 are the coördinates of any point, P'_0, on F_3 different from O, the projecting ray, OP'_0, is given in parameter representation by the equations,

$$(1) \qquad \rho\xi = t_1\xi_0,\ \rho\eta = t_1\eta_0,\ \rho\zeta = t_1\zeta_0,\ \rho T = t_1 T_0 + t_2.$$

The coördinates of the projection, P_0'', are

(2) $$\rho\xi=\xi_0,\ \rho\eta=\eta_0,\ \rho\zeta=\zeta_0,\ T=0.$$

The quantities ξ_0, η_0, ζ_0, which are not all three zero, are at the same time the trilinear coördinates* of the point, P_0'', with respect to the triangle of reference ABC in Π'. Only when the point, P_0', coincides with O does its projection become indeterminate. In order to avoid this exceptional case in the projection of K', we impose, upon the point O, *condition* 1 : *that the point O shall not lie on the curve K'.*

Vice versa: If ξ_0, η_0, ζ_0 are the trilinear coördinates of any point P_0'' in Π' with respect to the triangle of reference ABC, then the tetrahedral coördinates of the point P_0'' are ξ_0, η_0, ζ_0, 0, and the projecting ray OP''_0 is given by the equations,

(3) $$\rho\xi=\xi_0 t_1,\ \rho\eta=\eta_0 t_1,\ \rho\zeta=\zeta_0 t_1,\ \rho T=t_2.$$

If we substitute these values in the equation of the cubic surface, we will get a cubic equation in $t_1 : t_2$. If this cubic equation does not degenerate into an identity its three roots will give the parameters of the three points of intersection of the ray OP'' with F_3. One of these points is O, the two other points we denote by P_1' and P_2'. If the cubic equation degenerates into an identity, it is satisfied for all values of $t_1 : t_2$; and, therefore, the ray lies wholly on F_3. In order to avoid the exceptional case, we impose, upon the point O, *condition* 11 : *that the point O shall not lie on any one of the twenty-seven right lines on* F_3. We thus obtain a one-to-two correspondence of points between F_3 and Π'. The locus made up of the twenty-seven lines on F_3, we denote by Λ_1.

§11. *Related points.*

If λ_1°, μ_1°, ν_1° and λ_2°, μ_2°, ν_2° are the parameters of the two points P_1' and P_2' on F_3, and if P_1' does not coincide with O and does not lie on one of the six fundamental lines, then it can be shown that λ_2°, μ_2°, ν_2° are rationally expressible in terms of λ_1°, μ_1°, ν_1°; say,

(4) $\rho\lambda_2^\circ=\phi(\lambda_1^\circ,\mu_1^\circ,\nu_1^\circ)$, $\rho\mu_2^\circ=\psi(\lambda_1^\circ,\mu_1^\circ,\nu_1^\circ)$, $\rho\nu_2^\circ=X(\lambda_1^\circ,\mu_1^\circ\nu_1^\circ)$,

where ϕ,ψ,χ are homogeneous integral functions of $\lambda_1^\circ,\mu_1^\circ,\nu_1^\circ$.

§12. *One-to-one correspondence between the curves K' and K''.*

Although the projection from the point O establishes a one-to-two correspondence between the surface F_3 and the plane Π'; nevertheless, the correspondence between the curves K' and K'', defined by the same projection, is a one-to-one correspondence; provided the

*Clebsch-Lindemann, Vorlesungen über Geometrie, Bd. 11, p. 99.

curve K in the plane Π is *irreducible*, as we have supposed. In order to prove it, we consider any point P_1' of K' and ask under what conditions will the related point P_2' to P_1' also lie on K'?

We now make use of the hypothesis that K is *irreducible*; then it can be shown that there exists only a finite number m of pairs of related points, P_1' and P_2', which lie both on K'; the projections of two such related points coincide in a point of the curve K'' and give rise to a new multiple point of K''. These m new multiple points will be denoted by $D_1'', \ldots, D''_m$. To every other point of K'' corresponds one, and but one, point of K', and in this sense, *the projection from the point O establishes a one-to-one correspondence between the two curves K' and K''.*

§13. *The orders of the cycles of the curve K''.*

Let

$$(5)\qquad \begin{cases} \rho\xi=\xi_0+\xi_1t+\xi_2t^2+\ldots., \\ \rho\eta=\eta_0+\eta_1t+\eta_2t^2+\ldots., \\ \rho\zeta=\zeta_0+\zeta_1t+\zeta_2t^2+\ldots., \\ \rho T=T_0+T_1t+T_2t^2+\ldots., \end{cases} \qquad (C_0')$$

be a cycle of the curve K' expressed in the new system of coördinates. Since, according to §§7, 8, 9, all cycles of K' are linear we have,

$$(6)\qquad \left\|\begin{matrix} \xi_0 & \eta_0 & \zeta_0 & T_0 \\ \xi_1 & \eta_1 & \zeta_1 & T_1 \end{matrix}\right\| \neq 0,$$

where not all three of the quantities, ξ_0, η_0, ζ_0, are equal to zero, since the center of (C_0') is different from 0 (0,0,0,1), which is not on K'. The cycle (C_0') is projected into a new cycle (C_0'') in Π', given by the equations

$$(7)\qquad \begin{cases} \rho\xi=\xi_0+\xi_1t+\xi_2t^2+\ldots., \\ \rho\eta=\eta_0+\eta_1t+\eta_2t^2+\ldots., \\ \rho\zeta=\zeta_0+\zeta_1t+\zeta_2t^2+\ldots.. \end{cases} \qquad (C_0'')$$

We ask under what conditions will (C_0'') be of higher order than the first?

This will be the case if, and only if,

$$(8)\qquad \left\|\begin{matrix} \xi_0 & \eta_0 & \zeta_0 \\ \xi_1 & \eta_1 & \zeta_1 \end{matrix}\right\| = 0.$$

But if (8) is satisfied, we can determine two quantities k_1 and k_2, so that

$$(9)\quad O=\xi_0k_1+\xi_1k_2,\ O=\eta_0k_1+\eta_1k_2,\ O=\zeta k_1+\zeta k_2,$$

and therefore the point O lies on the tangent to the cycle (C_0') which may be written

$$(10)\qquad \begin{cases} \rho\xi=\xi_0t_1+\xi_1t_2, & \rho\eta=\eta_0t_1+\eta_1t_2, \\ \rho\zeta=\zeta_0t_1+\zeta_1t_2, & \rho T=T_0t_1+T_1t_2. \end{cases}$$

If, therefore, we construct, in every point P' of K', the tangent to K', or, in case P' should be a multiple point of K', the tangents to the linear cycles with coinciding centers at P' and denote by Q', in each case, the third point of intersection of the tangent with F_3; then, as the point P' describes the curve K' the point Q' will describe a curve on F_3, which we denote by Λ_2. If now we impose *condition* 111 upon the center of projection O, *that the point O shall be external to this locus* Λ_2, then the linear cycles of K' are projected into linear cycles of K'', and since all the cycles of K' are linear, also all the cycles of K'' will be linear.

§14. *The projection of the multiple points $A'_2, \ldots, A'_r$, of K'.*

Let (5) and

$$(11)\qquad \begin{cases} \rho\xi=\xi_0+\xi'_1t+\xi'_2t^2+\ldots, \\ \rho\eta=\eta_0+\eta'_1t+\eta'_2t^2+\ldots, \\ \rho\zeta=\zeta_0+\zeta_1't+\zeta'_2t^2+\ldots, \\ \rho T=T_0+T_1t+T_2t^2+\ldots. \end{cases}\qquad (C_1')$$

be two of the linear cycles with the multiple point A'_i of K' for center, then

$$\left\|\begin{matrix} \xi_0 & \eta_0 & \zeta_0 & T_0 \\ \xi'_1 & \eta'_1 & \zeta'_1 & T_1' \end{matrix}\right\| \neq 0;$$

and according to §8 (C_0') and (C_1'), have distinct tangents, that is

$$(13)\qquad \left\|\begin{matrix} \xi_0 & \eta_0 & \zeta_0 & T_0 \\ \xi_1 & \eta_1 & \zeta_1 & T_1 \\ \xi'_1 & \eta'_1 & \zeta'_1 & T_1' \end{matrix}\right\| \neq 0.$$

The new cycle (C_1'') of K'' is given by the equations

$$(14)\qquad \begin{cases} \rho\xi=\xi_0+\xi'_1t+\xi'_2t^2+\ldots, \\ \rho\eta=\eta_0+\eta'_1t+\eta'_2t^2+\ldots, \\ \rho\zeta=\zeta_0+\zeta'_1t+\zeta'_2t^2+\ldots. \end{cases}\qquad (C_1'')$$

The two cycles (C_0'') and (C_1'') will have coinciding tangents if, and only if

$$(15)\qquad \begin{vmatrix} \xi_0 & \eta_0 & \zeta_0 \\ \xi_1 & \eta_1 & \zeta_1 \\ \xi'_1 & \eta'_1 & \zeta'_1 \end{vmatrix} = 0.$$

This equation has a geometrical meaning: the equation of the plane passing through the tangents to (C_0') and (C_1') is according to §6,

$$(16)\qquad \begin{vmatrix} \xi & \eta & \zeta & T \\ \xi_0 & \eta_0 & \zeta_0 & T_0 \\ \xi_1 & \eta_1 & \zeta_1 & T_1 \\ \xi'_1 & \eta'_1 & \zeta'_1 & T_1' \end{vmatrix} = 0;$$

now if the point O lies in the plane (16), we have

$$(17)\qquad \begin{vmatrix} 0 & 0 & 0 & 1 \\ \xi_0 & \eta_0 & \zeta_0 & T_0 \\ \xi_1 & \eta_1 & \zeta_1 & T_1 \\ \xi'_1 & \eta'_1 & \zeta'_1 & T_1' \end{vmatrix} = 0,$$

that is

$$\begin{vmatrix} \xi_0 & \eta_0 & \zeta_0 \\ \xi_1 & \eta_1 & \zeta_1 \\ \xi'_1 & \eta'_1 & \zeta'_1 \end{vmatrix} = 0.$$

This plane (16) is the tangent plane to F_3 at P_0'; for the tangents to the cycles (C_0') and (C_1') are, at the same time, tangents to the surface F_3; and since F_3 has a definite tangent plane at A'_i (see §5), this tangent plane must be identical with the plane through the two distinct tangents to the cycles (C_0') and (C_1').

The plane (16) cuts the surface F_3 in a curve H. Hence, if we construct tangent planes to F_3 at each multiple point $A'_2, \ldots, A'_r$ of K', we shall obtain then $r-1$ curves $H_2, \ldots, H_r$, which constitute together a locus we denote by Λ_3; and if we impose, upon the center of projection O, *condition* IV, *that O shall be external to this locus,* then for each one of the multiple points $A'_2, \ldots, A'_r$ of K', *the σ_i linear cycles, $i=2,3,\ldots, r$, of A'_i will be projected into σ_i linear cycles with the same center A''_i and with distinct tangents.*

From the conditions $1-1V$ imposed so far upon the point O, it follows, therefore, that if P'' be any point of K'' different from the m points $D_1'', \ldots, D_m''$, then *either* P'' is the projection of an ordinary point of K', in which case P'' is again an ordinary point of K'', or P'' is the projection of one of the multiple points A'_i, $i=2,3,\ldots,r$, with σ_i linear cycles with distinct tangents, in which case P'' is an ordinary multiple point of K'' with the same number σ_i of linear cycles and with distinct tangents. It remains therefore only to study the points D'' of K''.

§15. *Condition to prevent the points D'' from being of higher multiplicity than the second order.*

In each point D'' coincide the projections of two related points P_1' and P_2' of K'. If the two points P_1' and P_2' are ordinary points of

K', the point D'' will be a double point of K''; if one of them is one of the multiple points A'_i, then the point D'' will be a multiple point of higher multiplicity.

The latter case will present itself if, and only if, the third point of intersection of the line OA'_i with the surface F_3 lies on the curve K'. Hence the projection of A'_i cannot coincide with the image of any other point P_0' of K', if the point O does not lie on the curve, we denote by Γ_i, described by the third point of intersection Q' of the A'_iP_0' with F_3 as the point P_0' describes the curve K'. If we denote by Λ_4 the locus made up of the $r-1$ curves $\Gamma_2, \ldots . \Gamma_r$ and impose *condition V* upon the point O, *that O shall be external to this locus* Λ_4, then the image of none of the multiple points $A'_2, \ldots .,A'_r$ will coincide with the image of any other point, simple or multiple, of the curve K'. *The points $D_1'', \ldots .,D_m''$ will therefore be double points of* K''; and on the other hand, if A'_i is the center of σ_i linear cycles of the curve K', then the image A_i'' of A_i', in the plane Π', will likewise be the center of σ_i (and not more) linear cycles.

§16. *Condition to prevent coinciding tangents at the double points $D_1'', \ldots, D_m''$.*

Let D'' be any one of the double points $D_1'', \ldots, D_m''$ and also $P_1'(\xi_1,\eta_1,\zeta_1,T_1)$ and $P_2'(\xi_2,\eta_2,\zeta_2,T_2)$ the two related points both lying on K', whose projections $P_1''(\xi_1,\eta_1,\zeta_1)$ and $P_2''(\xi_2,\eta_2,\zeta_2)$ in the plane Π' coincide in the point D'', so that

$$(19)\qquad \left\|\begin{matrix}\xi_1 & \eta_1 & \zeta_1\\ \xi_2 & \eta_2 & \zeta_2\end{matrix}\right\| = 0,$$

at the same time not all three of the quantities ξ_1,η_1,ζ_1 nor ξ_2,η_2,ζ_2 can be equal to zero, since on account of condition 1 neither P_1' nor P_2' coincides with O. Further the points P_1' and P_2' are distinct, according to §12, hence

$$(20)\qquad \left\|\begin{matrix}\xi_1 & \eta_1 & \zeta_1 & T_1\\ \xi_2 & \dagger_2 & \zeta_2 & T_2\end{matrix}\right\| \neq 0,$$

According to §15, P_1' and P_2' are both ordinary points of K'.

Let

$$(21)\qquad \begin{cases} \rho\xi = \xi_1 + \xi'_1 t + \ldots ., \\ \rho\eta = \eta_1 + \eta'_1 t + \ldots ., \\ \rho\zeta = \zeta_1 + \zeta_1 t + \ldots ., \\ \rho T = T_1 + T_1' t + \ldots ., \end{cases} \qquad (C_1')$$

and

$$(22)\qquad \begin{cases} \rho\xi = \xi_2 + \xi_2' t + \dots, \\ \rho\eta = \eta_2 + \dagger'_2 t + \dots, \\ \rho\zeta = \zeta_2 + \zeta'_2 t + \dots, \\ \rho T = T_2 + T_2' t + \dots, \end{cases} \qquad (C_2')$$

be the two linear cycles which represent the curve K' in the vicinity of P_1' and P_2' respectively.

The projections, in the plane Π', of these cycles are

$$(23)\qquad \begin{cases} \rho\xi = \xi_1 + \xi'_1 t + \dots, \\ \rho\eta = \eta_1 + \eta'_1 t + \dots, \\ \rho\zeta = \zeta_1 + \zeta'_1 t + \dots, \end{cases} \qquad (C_1'')$$

and

$$(24)\qquad \begin{cases} \rho\xi = \xi_2 + \xi'_2 t + \dots, \\ \rho\eta = \eta_2 + \eta'_2 t + \dots, \\ \rho\zeta = \zeta_2 + \zeta'_2 t + \dots, \end{cases} \qquad (C_2'')$$

respectively; their coinciding center is the point $\xi_1 : \eta_1 : \zeta_1 = \xi_2 : \eta_2 : \zeta_2$. The tangents to the cycles (C_1'') and (C_2'') will coincide if, and only if,

$$(25)\qquad \begin{vmatrix} \xi_1 & \eta_1 & \zeta_1 \\ \xi'_1 & \eta'_1 & \zeta'_1 \\ \xi'_2 & \eta'_2 & \zeta'_2 \end{vmatrix} = 0,$$

which has a simple geometric meaning. It easily follows that the tangents to the two cycles (C_1''), (C_2'') with the center D'' will coincide if, and only if, the tangents to the two cycles (C_1') and (C_2') lie in one plane passing through O.

We must now impose upon the point O, a condition which will prevent the point O from lying in a plane containing two tangents to K'. For this purpose, let P_1' be an ordinary plint of K', and let

$$(26)\qquad \alpha\bar{\xi} + \beta\bar{\eta} = 0$$

represent the pencil of planes through the tangent to K' at P_1'; $\bar{\xi}$, $\bar{\eta}$ being homogeneous linear functions of ξ, η, ζ, T. Let P_2' be another point of K', the image of an ordinary point P_2 of K; let

$$(27)\qquad \begin{cases} \lambda = \lambda_0 + \lambda_1 t + \lambda_2 t^2 + \dots, \\ \mu = \mu_0 + \mu_1 t + \mu_2 t^2 + \dots, \\ \nu = \nu_0 + \nu_1 t + \nu_2 t^2 + \dots, \end{cases}$$

be the linear cycle which represents the curve K in the vicinity of P_2. Then, it can be shown that there exists only *a finite number* of points P_2' on K' in which the tangent to K' lies in a plane of the pencil (26).

For each such point P_2' draw the line $P_1' P_2'$, it meets the surface F_3 in a third point R. As the point P_1' describes the curve K', these R points will describe a finite number of curves on F_3. The totality of these curves makes up a locus which we denote by Λ_5. Hence, if we impose upon the center of projection O, *condition* V1, *that* O *shall not lie on this locus* Λ_5, then *the new double points* $D_1'', \ldots, D_m''$ will have distinct tangents.

We now apply to the curve K'' and to the multiple point A_2'' the same process which has been applied to the curve K and the multiple point A_1, and so on. We repeat the process as many times as there are multiple points to be resolved, taking each time the new curve as the curve to be transformed, and finally a curve will be reached possessing a finite number of ordinary double points with distinct tangents, but no other multiple points.

Combining this solution with *Noether's Theorem* previously mentioned, we have the final theorem that ***every algebraic curve can be transformed by a birational transformation of the curve into a curve which has no other multiple points but ordinary double points with distinct tangents.***

Humanism and History of Mathematics

Edited by
G. WALDO DUNNINGTON

Carl Friedrich Geiser

By ARNOLD EMCH
University of Illinois

1. The passing of Geiser marks the end of a great period of algebraic geometry, the old synthetic school initiated by Poncelet, Steiner and Chasles. The influence of Steiner in particular has contributed considerably in starting such men as Cremona, Sturm, Schroeter, Reye and Geiser on a successful career as geometricians.

2. Carl Friedrich Geiser was born February 26, 1843, in Langenthal, Canton of Berne, Switzerland, where his father was an innkeeper, butcher and member of the Cantonal legislature at Berne. Jakob Steiner was his great-uncle who urged him to study mathematics. He studied four semesters at the Federal Polytechnic School in Zurich (now Eidgenössische Technische Hochschule) and then four semesters under Weierstrass and Kronecker at the University of Berlin. As Geiser was without sufficient means from home he had to earn his living by giving private lessons which were procured for him by the two great Berlin mathematicians. In 1863 he returned to Zurich to act as a substitute for the chair of descriptive geometry, vacated by the death of Deschwanden. This position was later filled by the appointment of W. Fiedler. During this time Geiser received the doctor's degree and in 1873 he was appointed to a full professorship of mathematics for engineering and mathematics students. His chief interest was in courses for mathematicians. They comprised the theory of algebraic curves and surfaces, invariants and covariants, and differential geometry. It was in the early period of his activity at Zurich that Geiser accomplished his important researches in algebraic geometry. He was rather exacting in his expectations from his students and, for this reason, was not always very popular, especially among the engineers.

In the course of years Geiser developed valuable administrative abilities and for ten years performed the duties of director of the Polytechnic. He was the confidential adviser of the Federal School

Carl Friedrich Geiser

Board (Eidgenössischer Schulrat) and its president, Mr. Kappeler. Under this administration the Swiss institution became one of the most outstanding schools of its kind in the world. During this period we find among the mathematicians who taught at Zurich such men as Dedekind, Durège, Christoffel, Schwarz, Weber, Reye, Fiedler, Frobenius, Schottky, Hurwitz, Minkowski and Zermelo. Other departments were equally well staffed with outstanding men. To add a personal note I may say that I had the good fortune to have Geiser, Fiedler, Frobenius, Schottky and Hurwitz as my teachers and to get at least a touch of the inspiration that emanated from those splendid men.

In the educational field Geiser was very influential in bringing the Cantonal (or secondary) colleges of Switzerland to a higher level of teaching and of professional qualifications for teachers of those schools. In 1897, when the first International Congress of Mathematicians met at Zurich, Geiser was elected president of it. Although a good European in the best sense of the word, Geiser remained a rugged individualist of the old Swiss type. During this Congress a lecture by Reye was scheduled to take place in one of the auditoriums of the Polytechnic. We who were specially interested in algebraic geometry, entered the room to get a good seat. The lecture began promptly. But it seemed that everybody wanted to hear Reye. So the crowd continued to increase till the walls were lined with spectators. Finally the stragglers became so annoying that Geiser with the back of his tall body leaned against the door to prevent further disturbance. He hated all pretense and always worked in support of real merit. True friendship was guarded as a treasure by him. Having been an intimate friend of the famous Swiss writer Gottfried Keller, Geiser was asked to contribute to a collection of anecdotes about the original character of the poet. Geiser flatly refused to be a party to such a questionable enterprise.

Although struck by loss of clear eyesight and temporary blindness which was later partly removed by a surgical operation, Geiser's mind remained active and alert. When 89 years of age, being especially fond of Italian culture, he insisted on being taken to Zurich from his Chalet on the lovely mountain-side of Lake Zurich to hear a lecture on Dante by the Swiss-Italian poet Chiesa.

The writer last saw Geiser in the spring of 1929, when he spent a friendly afternoon with him at his home on the lake. He died, aged 91, on March 7, 1934.

3. In the obituary* of Geiser by Professor Scherrer of Zurich, following that of Professor Meissner, the list of scientific entries papers

Vierteljahrschrift der Naturforschenden Gesellschaft in Zürich, Vol. 69 (1934), pp. 371-376.

and addresses contains forty entries. In this short biographical sketch I shall mention only a few of the most notable contributions by Geiser to algebraic geometry.

One of the earliest papers appeared in 1865* on a certain quadratic transformation, which was also found in the same year by T. A. Hirst.† It simply consists of this: Given a conic C and a fixed point A_1 in the same plane but not on C. Let P be a generic point in the plane. Join P to A by a line s, and construct the polar p of P with respect to C. The intersection P' of p with s corresponds to P in the so-called *quadric inversion.* The intersections A_2 and A_3 of the polar a_1 of A_1 with C and A_1 are the fundamental points of this transformation.

Two years later appeared a paper on *Zwei Geometrische Probleme*‡ concerned with involutorial transformations. The first deals with a net of cubics on seven generic points $A_1\ A_2\ \cdots A_7$ in a plane. All cubics of the net that pass through a generic point P pass through another point P'. The relation between P and P' is an involutorial transformation of order s. $A_1\ A_2\ \cdots A_7$ are fundamental points and the cubics through six of these and with a double point at the seventh as fundamental curves. There exists a pointwise invariant curve $(P \equiv P')$ C_6 with the A's as double points and with tangents at these which are identical with those at the double points of the corresponding fundamental curves. This sextic of genus three is sometimes called the Aronhold sextic.

The second problem is concerned with a web of quadrics on six generic points $A_1\ A_2\ \cdots A_6$ in S_3. All quadrics of the web that pass through a generic point P pass through another point P', determined by P. The correspondence is an involutorial transformation of order seven in S_3. $A_1\ A_2\ \cdots A_6$ are fundamental points and the quadric cones on five of these with the sixth as a vertex are fundamental surfaces. There exists a pointwise invariant quartic surface with the A's as double points. This is the well known Weddle surface W. The cubic determined by the A's lies on W. These transformations are generally known today as *Geiser transformations.*

It is interesting to know that at the time of this publication several years had passed since Cremona's classic investigations on general transformations which now go by *Cremona's* name. It is possible that Geiser was not aware of these discoveries.

In 1869 Geiser published another important paper|| on the double tangents of quartic plane curves, in which he showed the connection

**Mitteilungen der Berner Naturforschenden Gesellschaft* No. 592, pp. 97-107.
†*Royal Society Proceedings*, Vol. 14, pp. 91-106.
‡*Journal für reine und angewandte Mathematik*, Vol. 67 (1867), pp. 78-89.
||*Mathematische Annalen*, Vol. 1 (1869), pp. 129-138.

between these and the 27 lines of a general cubic surface F_3. Choose a generic point O on F_3 and construct the residual quartic tangent cone K_4 from O to F_3. Let α_{28} be the tangent plane to F_3 at O, and $\alpha_1, \alpha_2, \cdots, \alpha_{27}$ the planes joining O to the 27 lines of F_3. A generic plane α cuts K_4 in a plane quartic C_4, and $\alpha_1, \alpha_2, \cdots, \alpha_{27}, \alpha_{28}$ in 28 lines which are the double tangents of C_4.

The Teacher's Department

Edited by
JOSEPH SEIDLIN and JAMES McGIFFERT

A Fresh Start

By EDWIN G. OLDS
Carnegie Institute of Technology

Introduction. The second semester began today and, with it came the challenge—a class of "repeaters" in Differential Calculus. Not a new challenge for, each year, throughout the collegiate world, boys "flunk" Calculus and then have to repeat it.

"Why talk about it?", you ask? Because I'd like to tell you what we did today and get your reaction to it, based on your experience in similar situations.

At the bell the room was only half full but the back row was solid. Students in this row were ushered to more desirable seats. Late comers took the vacated seats, were invited to move, and saw later comers fill the seats again. Somehow it seemed that they didn't care to get too close to Calculus.

Most of the names and faces were all too familiar. Like the "March of Time", past records flashed across the screen of memory. But memory was unnecessary to predict their attitude, their appearance of complete boredom did that. Not a flicker of interest pierced the wintry gloom.

Probably it was a bad beginning to mention past records but it seemed necessary to announce that they would not count against anyone. This provided the proper opening for the statement that we (the class) were going to discuss Calculus together just as though we knew nothing about it—and we did just that. However, the teacher, at least, added the mental note that we didn't have any very perfect understanding of the prerequisite mathematics either.

The Lesson. Differential Calculus is different from any of the mathematics you have studied before, because you can learn how to perform the manipulations without really learning any Calculus at all. (You, the readers, are now getting the lecture as the students

heard it. You recognize that the first statement is highly inaccurate. Of course Calculus is not "different" in the respect noted.) You can work the assigned problems in a few minutes; the important part of your preparation is the time you spend in thinking about the principles involved. But do not be alarmed. Differential Calculus is actually quite easy because we have only one idea to gain, we're interested in the derivative and that's all. We start talking about it the first day and study it the whole semester. There is none of this jumping around from one topic to another and not understanding much about any of them.

Let x represent the length of the edge of a cubical vessel, filled with water; let w represent the weight of a cubic unit of water, and let y be the weight of the water in the vessel.

Is there any connection between these quantities? (Some one is found who states that $y=wx^3$.)

What is the difference between w and x? (w is a constant, x is a variable.)

How about y? (y is also a variable.)

Is it the same kind of a variable as x? (No, y is a dependent variable, x is independent.)

What do we call a relationship of this sort? (A functional relationship.)

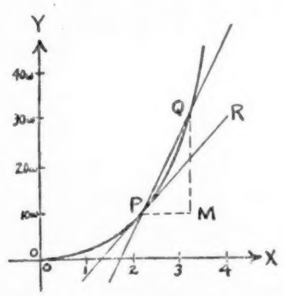
Can we express x as a function of y? $\left[\text{Yes, } x=\sqrt[3]{\frac{y}{w}}.\right]$

How is this related to $y=wx^3$? (The two are inverse functions.)

How can we picture the relation between y and x? (By means of a graph.)

(We proceed, then, to prepare a table of values and sketch the curve, noticing that it is convenient to scale the y–axis in terms of w.)

Δx	x	y	Δy
	0	0	
1			w
	1	w	
1			7w
	2	8w	
1			19w
	3	27w	
1			37w
	4	64w	



Let's look at our table of values again. We have changed x one unit at a time. Most of you are familiar with the notation, Δx, for this.

Δx is called, also, the increment of x. Then we may denote the increment of y by Δy. Computing the values for Δy we notice that, in spite of the fact that all the increments of x are the same, the corresponding increments of y are different. So Δy must depend on something beside Δx, because, if Δy depends only on Δx, then, when two values of Δx are equal, the corresponding values of Δy would be equal.

(This provides an opening to suggest the computation of a general expression for Δy.)

$$y_1 = wx_1^3$$

$$y_1+\Delta y_1 = w(x_1+\Delta x)^3$$

$$\Delta y_1 = w(3x_1^2\Delta x+3x_1\overline{\Delta x}^2+\overline{\Delta x}^3)$$

Let's check the value of $19w$ in our table of values. This corresponds to x equals what? (two) and Δx equals what? (one). Substituting, we find

$$\Delta y_1 = w(12+6+1) = 19w$$

as we expected.

Examining the general expression for Δy, of what is Δy a function? (At this point the discovery is made that several people cannot answer because they do not know what is meant by "function". The remarks made at this juncture are omitted.) Since Δy depends on both x and Δx we can see why there appeared different values of Δy corresponding to equal values of Δx.

Do you think of a function where Δy depends on Δx only? (Some one suggests a linear function.) Yes, if we write

$$y = mx+b, \text{ (for } m \text{ and } b \text{ fixed)},$$

then $$y+\Delta y = m(x+\Delta x)+b$$

and $$\Delta y = m\Delta x.$$

What is m? (It is the slope of the line.)

Then $$\frac{\Delta y}{\Delta x} = m.$$

(We proceed to draw a line and use it to show how for any x, the ratio of Δy to Δx is constant.)

How about seeing what the situation is for our curve?

(We locate two points, $P(x,y)$ and $Q(x+\Delta x,\ y+\Delta y)$, on the cubic. Somebody hazards a guess that $\Delta y \div \Delta x$ is the slope of the tangent at P. When this tangent is drawn he changes his mind. Finally the proper answer is given. But the tangent line is on the graph and it's

easy to direct attention to it and lead the students to see that the secant will rotate toward the tangent if we decrease Δx. Then, returning to the algebraic expression for Δy, we divide by Δx, take the limit as Δx approaches zero, and define the derivative. Thus we keep our promise by talking about the derivative in the first lesson. Also we are able to give it a meaning.)

Conclusion. Tomorrow we'll review much of what we did today and, I hope, find time to talk about rate of change. Next week, we'll discuss the same things again, but in less detail, and with the students doing the lion's share of the work. Each day we'll try to clarify some elementary concept. Before the end of the semester some of the boys may lose their air of boredom. Perhaps a few will develop the mathematical power which comes only as a result of meaningful appreciation.

On Euler's Forms

By JOS. B. REYNOLDS
Lehigh University

The usefulness of Euler's forms for the sine and cosine of an angle in integrating even powers of these functions and in other cases makes an elementary derivation of these forms a desirable teaching device. It is the purpose of this article to present two such derivations.

(1) Suppose

$$y = \cos\Theta + i\sin\Theta,$$

then

$$\frac{dy}{d\Theta} = -\sin\Theta + i\cos\Theta = iy$$

whence

$$dy/y = i d\Theta$$

and, therefore

$$\log y = i\,\Theta + C.$$

Now $C = 0$, since $y = 1$ for $\Theta = 0$, which gives

$$y = \cos\Theta + i\sin\Theta = e^{i\Theta}.$$

Upon taking the reciprocal of both sides we find

$$\cos\Theta - i\sin\Theta = e^{-i\Theta},$$

whence

$$\sin\Theta = (e^{i\Theta} - e^{-i\Theta})/2i \text{ and } \cos\Theta = (e^{i\Theta} + e^{-i\Theta})/2.$$

(2) Upon integrating both sides of the identity

$$-du/(1-u^2)^{\frac{1}{2}} = -i\,du/(u^2-1)^{\frac{1}{2}}$$

we obtain

$$\cos^{-1}u = -i\log(u - (u^2-1)^{\frac{1}{2}}) + C.$$

By substituting $u = 1$ we find $C = 0$.

Now, letting $\cos^{-1}u = \Theta$, we have

$u = \cos\Theta$ and $(u^2-1)^{\frac{1}{2}} = i\sin\Theta$, so that

$$i\Theta = \log(\cos\Theta + i\sin\Theta)$$

and, as before

$$\cos\Theta + i\sin\Theta = e^{i\Theta}.$$

As an illustration of the use of these forms in elementary mathematics consider the following:

$$\int \sin^6\theta \, d\theta = \int [(e^{i\theta} - e^{-i\theta})/2i]^6 d\theta$$

and upon expanding we find, after integrating the righthand member,

$$-64 \int \sin^6\theta \, d\theta$$

$$= (e^{6i\theta} - e^{-6i\theta})/6i - 3(e^{4i\theta} - e^{-4i\theta})/2i + 15 \left(\frac{e^{2i\theta} - e^{-2i\theta}}{2i} \right) - 20\theta$$

$$= \sin 6\theta/3 - \sin 4\theta + 15 \sin 2\theta - 20\theta;$$

so that finally we have

$$\int \sin^6\theta \, d\theta = (60\theta + 45 \sin 2\theta - 9 \sin 4\theta - \sin 6\theta)/192 + C.$$

The General Solution of the Exact Differential Equation $Mdx+Ndy=0$

By V. W. ADKISSON
University of Arkansas

We propose the following proof that

$$\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$$

be a sufficient condition for $Mdx+Ndy=0$ to be exact, and from this proof obtain a very satisfactory working rule for solving this type of differential equation.

Let

$$\int^x Mdx=f(x,y)+F(y), \qquad (y, \text{constant})$$

$$\int^y Ndy=g(x,y)+G(x), \qquad (x, \text{constant})$$

where regardless of how f is expressed every term contains x, and regardless of how g is expressed every term contains y. Then

$$\frac{\partial}{\partial x}(f-g)=M-\frac{\partial}{\partial x}\int^y Ndy+G'(x)=\text{function of } x \text{ only},$$

since

$$\frac{\partial^2}{\partial y\,\partial x}(f-g)=0.$$

Hence, $(f-g)$ has no terms involving both x and y.

Let

$$f=v(x,y)+h(x)$$

$$g=v(x,y)+k(y)$$

where the terms in v involve both x and y. Then

$$u(x,y)=v(x,y)+h(x)+k(y)=\text{constant}$$

is the general solution of $Mdx+Ndy=0$. For

$$\frac{\partial u}{\partial x}=\frac{\partial v}{\partial x}+h'(x)=\frac{\partial f}{\partial x}=M, \text{ and}$$

$$\frac{\partial u}{\partial y}=\frac{\partial v}{\partial y}+k'(y)=\frac{\partial g}{\partial y}=N.$$

The rule may now be stated as follows: Obtain the terms of $\int^x Mdx$ that involve x, and the terms of $\int^y Ndy$ that contain y but not x, and equate the sum of these terms to a constant.

This rule is easily applied, and avoids certain difficulties which may otherwise arise. See Conkwright, *Differential Equations*, p. 18.

Mathematical World News

Edited by
L. J. ADAMS

The American Mathematical Society will hold a sectional meeting at the University of Michigan, Ann Arbor, on April 8-9, 1938. Professor C. G. Latimer will deliver an address entitled *The arithmetics of generalized quaternions*. Another sectional meeting will be held at Charlottesville, Virginia on April 15-16, 1938. The program will include: *Applications of Lattice Theory* by Mr. Garrett Birkhoff, *Structures and Algebraic Applications* by Professor Oystein Ore, and *Representations of Boolean Algebras* by Professor M. H. Stone.

The quarterly review, *Science Progress*, published in London by Edward Arnold and Co., has a department devoted to Recent Advances in Science, in which mathematics occupies one section. An example of the type of article found there is the one on *The Mathematical Theory of Knots* by J. A. C. Whitehead in the July, 1937 issue. The article is expository and historical.

On October 18-21, 1937 there was held in Bologne, Italy, a *Congrès de Physique*, organized and held to celebrate the second centennial of the birth of Luigi Galvani. Some of the foreign scientists present were Bohr, Heisenberg, Schrödinger, Debye, Sommerfeld, Richardson, Pauli, and De Broglie.

Mr. P. Hall has been appointed secretary of the London Mathematical Society, and as such will receive manuscripts for publication in the *Proceedings* of the Society.

The *Journal of Calendar Reform* is published by The World Calendar Association, Inc., an organization devoted to the adoption of a calendar planned to eliminate certain objections to the calendar now in common use.

An interesting addition to the literature on mathematical recreations is *La Mathématique des Jeux*, by M. Kraitchik, editor of *Sphinx*. The volume has 566 pages and is divided into three parts: Jeux de calcul, Jeux de position and Jeux de permutation.

The first two volumes of *Foundations of the Unity of Science* have been announced for publication during 1938 and 1939. They will be

the general introduction to the International Encylopedia of Unified Science. The article on *Mathematics and Logic* will be written by Rudolph Carnap.

Zentralblatt für Mathematik und Ihre Grenzgebiete contains reviews of current articles selected from journals all over the world. The reviews are grouped by topics, such as philosophy, logic, algebra and number theory, analysis, group theory, geometry and so forth. It is published by the firm of Julis Springer in Berlin.

Professor T. C. Fry, Bell Telephone Laboratories, was scheduled to deliver two addresses at the University of California, Los Angeles, on February 21 and 23, respectively.

The annual list of published papers by members of the American Mathematical Society appears as a supplement to the January issue of the *Bulletin* of the Society.

Professor Tomlinson Fort is now in charge of book reviews for the *Bulletin* of the American Mathematical Society, and Professor G. A. Hedlund is the new assistant editor for the same periodical.

G. E. Stechert and Co. import foreign books on mathematics through their offices in Leipzig, London and Paris. Among their offerings is a series of seven volumes of the Polish *Monografic Matematyczne*, printed in Warsaw. Since the publications of Polish mathematicians have come into particular prominence in recent years, we publish the list of titles of six of these seven volumes here:

1. *Theorie des operations lineaires.* S. Banach.
2. *Topologie.* C. Kuratowski.
3. *Hypothese du continu.* W. Sierpinski.
4. *Trigonometrical Series.* A. Zygmund.
5. *Theorie der Orthogonalreihen.* S. and H. Steinhaus.
6. *Theory of the Integral.* Stanlislaw Saks. Translated into English by L. C. Young.

Dr. József Jelitai of Budapest, Hungary, presented on December 13, 1937, to Class III of the Hungarian Academy of Sciences a paper entitled *Letters from Gauss and Encke in the Hungarian Provincial Archive.* Dr. Jelitai is preparing three articles on Pühler's Geometry of 1563, with special reference to its significance in astronomy, geodesy and the measurement of depths. In February, 1938, he presented to the above mentioned section of the Hungarian Academy a paper *Letters of Daniel Bernouli and Clairaut to Count Josef Teleki,* and in the same month gave several lectures before the astronomy section of the

Hungarian Natural History Society on *New Archive Data Concerning the History of Astronomy in Hungary Between 1774 and 1843.* His most recent publications are *Clairaut, la Condamine, d'Alembert and their Contemporaries According to the Diaries of Counts Josef and Samuel Teleki,* in Mat. és. Fizikai Lapok. 44. 1937. pp. 173-199, and *Wolfgang von Bolyai and the Institutum Pensionale Hungaricum,* in Mat. és. Fizikai Lapok. 44. 1937. pp. 168-172. In addition to the above Dr. Jelitai is preparing to edit the geometrical manuscript of the Hungarian Provincial Archive; it is written in Old French, dated ca. 1600, and occurs in the *Liber Regius von Bethlen.* In the summer of 1938 he plans to collect Bolyai material in the home of these mathematicians at Marosvásárhely, which is now called Targu-Mures, and is in Rumania.

Problem Department

Edited by
ROBERT C. YATES and EMORY P. STARKE

This department solicits the proposal and solution of problems by its readers, whether subscribers or not. Problems leading to new results and opening new fields of interest are especially desired and, naturally, will be given preference over those to be found in ordinary textbooks. The contributor is asked to supply with his proposals any information that will assist the editors. It is desirable that manuscript be typewritten with double spacing. Send all communications to Robert C. Yates, College Park, Maryland.

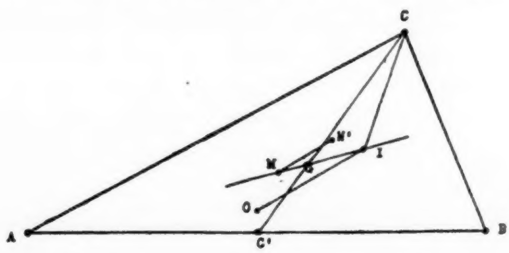
SOLUTIONS

No. 129. Proposed by *Walter B. Clarke*, San Jose, Calif.

In the triangle ABC, I is the incenter, O is the circumcenter, N' is the center of the nine-point circle, C' is the midpoint of AB and M is the concurrent point of the lines from the midpoints of the sides to the point half way around the perimeter. Show that

$$MN'/MC' = IO/IC.$$

Solution by *Karleton W. Crain*, Purdue University.



G, the centroid, is on the line segment IM, and $IG/GM = 2/1$. (Cf. solution to problem E 131, *The American Mathematical Monthly*, Vol. XLII, 1935, page 393.) $CG/GC' = 2/1$, since the centroid trisects each median, also $OG/GN' = 2/1$. (Cf. Johnson, *Modern Geometry*, page 165, paragraphs 257, 258.)

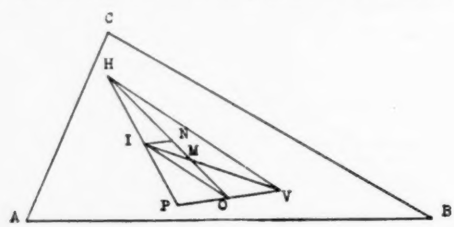
Therefore, G is the center of similitude for the two similar triangles OIC and $N'MC'$, and $MN'/MC' = IO/IC$. Since the ratio of similitude is $2/1$, we may write $IC/MC' = IO/MN' = 2/1$.

No. 170. Proposed by *Walter B. Clarke*, San Jose, Calif.

Using the notation, I for incenter, O for circumcenter, H for orthocenter, V for verbicenter, R for circumradius, r for inradius, of a plane triangle, show that

$$(IV)^2+(IH)^2=4(OV)(R-r).$$

Solution by *Karleton W. Crain*, Purdue University.



Let M be the centroid, the point of intersection of HO and IV. (Note, the verbicenter is called Nagel point by Johnson, *Modern Geometry*, pp. 149, 225.) Now in triangles OIV and HVI,

(1) $$(OV)^2=(OI)^2+(IV)^2-2(OI)(IV)\cos\angle OIV \text{ and}$$

(2) $$(IH)^2=(HV)^2+(IV)^2-2(HV)(IV)\cos\angle HVI.$$

But $\angle OIV = \angle HVI$, since HV is parallel to IO.*
Since $HV=2(OI)$,† if equation (1) is multiplied by two, and subtracted from equation (2), we have $(IV)^2+(IH)^2=2(OV)^2+2(OI)^2$. But $(OI)^2=R(R-2r)$ and $OV=2(IN)=(R-2r)$.‡ [For if OV and IH are extended so as to intersect at P, then O and I are the midpoints of VP and HP. Then as N is the midpoint of OH, IN is parallel to PO and $IN=PO/2=OV/2$.] Therefore, $(IV)^2+(IH)^2=2(R-2r)^2 +2R(R-2r)=4(R-2r)(R-r)=4(OV)(R-r)$.

No. 184. Proposed by *Albert Farnell*, Centenary College.

Integrate: $$\int \sec^4 x\cdot\tan x\cdot\log^4\sec x\cdot dx.$$

*Johnson, *Modern Geometry*, p. 226.
†Ibid., p. 226.
‡Ibid., p. 205.

Solution by *M. L. Vest,* Davis and Elkins College.

This can be solved very simply by using iterated integration by parts. Setting

$$u=\log^4\sec x, \quad dv=\sec^4 x\cdot\tan x\cdot dx$$

the value of the integral becomes

$$\tfrac{1}{4}\sec^4 x\cdot\log^4\sec x-\int\sec^4 x\cdot\tan x\cdot\log^3\sec x\cdot dx.$$

Here it is seen that the power of the logarithm has been reduced by one, while the remainder of the integrand remains the same. Hence we integrate three more times, using a similar division into parts each time. This gives the result of the integration as

$$\frac{1}{128}\sec^4 x\Big[32\log^4\sec x-32\log^3\sec x+24\log^2\sec x-12\log\sec x+3\Big]+C$$

Also solved by *Dewey C. Duncan, Leo D. Simmons, C. W. Trigg, Yudell Luke, W. T. Short,* and the *Proposer.*

No. 185. Proposed by *V. Thébault,* Le Mans, France.

With the digits 0,1,2,3,4,5,6,7,8,9 taken once, form a perfect square which is divisible by 66. The solution is unique.

Solution by *C. W. Trigg,* Eagle Rock H. S., Los Angeles.

Let $N=M^2$ be the desired square. If $N\equiv 0 \pmod{2\cdot 3\cdot 11}$ we must have also $M\equiv 0 \pmod{66}$. Since $1023456789\leqq N\leqq 987654321$, it follows that $32010\leqq M\leqq 99330$. From a table of the squares of the numbers less than 10,000 may be read the last four and, in general, the first three digits of N corresponding to the multiples of 66 within the range established for M. Appearance of duplicate digits in these parts of N reduces the possible values of M to thirty-nine. When these are squared, the following four solutions are revealed:

$$39336^2=1547320896, \qquad 60984^2=3710948256,$$

$$98802^2=9761835204, \qquad 99066^2=9814072356.$$

Thus the solution is not unique.

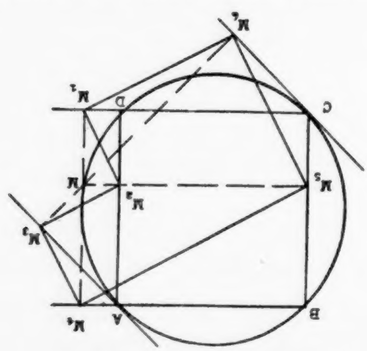
(In problem E 291, *American Mathematical Monthly,* it is shown that $19569^2 = 382945761$ is unique for N divisible by 99. Hence there exists no N composed of the digits exclusive of zero and divisible by 66.)

Also solved in part by *F. M. Kenny.*

No. 186. Proposed by *V. Thébault,* Le Mans, France.

Consider the square $ABCD$ and the tangents to the circumcircle at two opposite vertices. $M_1, \cdots\cdots, M_6$ are the orthogonal projections of an arbitrary point, M, of this circle upon the sides of the square and the two tangents. Show that $M_1, \cdots\cdots, M_6$ are the vertices of a hexagon of area equal to that of the square and whose consecutive sides are perpendicular.

Solution by *Karleton W. Crain,* Purdue University.



Let $A(1,1)$, $B(-1,1)$, $C(-1,-1)$, $D(1,-1)$ be the rectangular coordinates of the square. The equation of the circumscribing circle is $x^2+y^2=2$. The equations of the tangents to it at A and C are $x+y-2=0$ and $x+y+2=0$. The projections of $M(x',y')$, a point on the circle, upon the sides of the square and the two tangents have the coordinates $M_1(x',-1)$, $M_2(1,y')$,

$$M_3\left(1+\frac{x'-y'}{2}, 1+\frac{y'-x'}{2}\right), M_4(x',1), M_5(-1,y'),$$

$$M_6\left(-1+\frac{x'-y'}{2}, -1+\frac{y'-x'}{2}\right)$$

where these points are taken in counter-clockwise order. Using these six sets of coordinates and the determinant method, we can easily

reduce the area of this hexagon to four sets by using the identical relation $x'^2+y'^2=2$.

Likewise by the aid of this same relation, the slopes of M_1M_2, M_3M_4, M_5M_6 are equal and are the negative reciprocals of the slopes of M_6M_1, M_2M_3, and M_5M_4. Therefore, the consecutive sides of the hexagon are perpendicular.

Also solved by *Walter B. Clarke*, *W. T. Short*, and the *Proposer*.

No. 188. Proposed by *V. Thébault*, Le Mans, France.

In what number system can a number of four digits of the form *aabc* be the square of a number of two digits, *mn*, being given that $c=b+1$ and $m=n+1$?

Solution by the *Proposer*.

The solution of this interesting question falls at once into three cases, B being the desired base:

(1) $(B+1)=a+b+2(n+1)$ and $(n+1)^2=a(B-1)+1$;

Example, $B=17$, $3312=76^2$.

(2) $2(B+1)=a+b+2(n+1)$ and $(n+1)^2=a(B-1)+2$;

Example, $B=15$, $7756=(\overline{10}\ 9)^2$.

(3) $3(B+1)=a+b+2(n+1)$ and $(n+1)^2=a(B-1)+3$;

Example, $B=34$, $\overline{22}\ \overline{22}\ \overline{29}\ \overline{30}=(\overline{27}\ \overline{26})^2$.

G. W. Wishard finds, as under (1), that if $B=m^2$ we always have 1, 1, n^2-1, n^2 as the digits of the square of *mn*. *C. W. Trigg* notes also that 15 is the smallest value of B for which the letters represent distinct digits.

Editor's Note: A basis for the above solution may be supplied as follows. The given problem implies the equation

$$aB^3+aB^2+bB+c=(mB+n)^2$$

If we substitute $c=b+1$ and $m=n+1$, transpose the term 1 and divide by $B+1$, we have the condition for a solution in the form

$$aB^2+b=(n+1)^2B+n^2-1.$$

This is equivalent to the simultaneous equations with parameter x.

$$(n+1)^2=a(B-1)+x \text{ and } x(B+1)=a+b+2(n+1)$$

Since a, b and $m=n+1$ are digits less than B, the second of these equations requires $x(B+1)<4B$. Thus $x=1, 2, 3$.

No. 189. Proposed by *J. Rosenbaum*, Bloomfield, Connecticut.

Prove that if $x^3+y^3=z^3$ has a solution in integers, all different from zero, then there exist integral solutions, all different from each other and from zero, for the following:

(a) sum of three cubes equals the sum of two cubes.
(b) sum of four cubes equals the sum of three cubes.
(c) sum of five cubes equals the sum of four cubes.

Solution by *Harry M. Gehman*, University of Buffalo.

The following generalization may be proved:

Theorem: Let $F(x)=x^m$, where m is a constant, not necessarily integral, and suppose that the equation

$$F(x_1)+F(x_2)=F(y_1) \tag{1}$$

has a solution $x_1=a$, $x_2=b$, $y_1=c$, where a, b, c are different from zero and are such that no relation exists of any of the forms:

$$a^g b^h c^j=1, \quad a^g b^h=1, \quad a^g c^j=1, \quad b^h c^j=1, \tag{2}$$

where g, h, j are positive or negative integers. Then for any positive integral value of n, the equation

$$\sum_{i=1}^{n+1} F(x_i) = \sum_{i=1}^{n} F(y_i) \tag{3}$$

has a solution in which the values of the unknowns are different from each other and from zero. In particular, if (1) has a solution in integers, then (3) has a solution in integers.

Proof: By hypothesis (3) has a solution of the required form when $n=1$. We shall assume that (3) has a solution of the required form when $n=r$; i. e., there exists a set of constants $d_1, \cdots, d_{r+1}; e_1, \cdots, e_r$, such that

$$\sum_{i=1}^{r+1} F(d_i) = \sum_{i=1}^{r} F(e_i). \tag{4}$$

The result of multiplying (4) by a^{2m} may be put in the form

$$\sum_{i=1}^{r} F(a^2 d_i) + F(a^2 d_{r+1}) = \sum_{i=1}^{r} F(a^2 e_i). \tag{5}$$

Multiplying $F(a)+F(b)=F(c)$ by $b^m d_{r+1}^m$ and $a^m d_{r+1}^m$, we have respectively

(6) $$F(abd_{r+1})+F(b^2d_{r+1})=F(bcd_{r+1}),$$

(7) $$F(a^2d_{r+1})+F(abd_{r+1})=F(acd_{r+1}).$$

By use of (7), the sum of (5) and (6) may be written

$$\sum_{i=1}^{r} F(a^2d_i)+F(acd_{r+1})+F(b^2d_{r+1})=\sum_{i=1}^{r} F(a^2e_i)+F(bcd_{r+1}).$$

Thus, if $d_1,\cdots,d_{r+1}$; $e_1,\cdots,e_r$ is a solution of (3) when $n=r$, then $a^2d_1,\cdots,a^2d_r$, acd_{r+1}, b^2d_{r+1}; $a^2e_1,\cdots a^2e_r$, bcd_{r+1} is a solution of (3) when $n=r+1$. In this solution each unknown is different from zero and no two are equal, for otherwise we would have a relation of type (2) contrary to hypothesis.

In the proposed problem, $m=3$; $n=2$, 3 4; a, b, c, integers, it may easily be shown, by theorems on rational roots of a polynomial equation, that no two values obtained are equal. Hence in this special case it is necessary to impose the restriction (2).

D. C. Duncan obtains the same solution (for $m=3$) by multiplying both members of $x^3+y^3=z^3$ by $\sum_{i=0}^{k}(-1)^i x^{3(k-i)}y^{3i}$ and transposing all negative terms. For $k=2r-1$, he obtains the sum of $r+1$ cubes equal to the sum of $r+1$ cubes; but for $k=2r$, the sum of $r+2$ cubes equal to the sum of $r+1$ cubes.

The *Proposer's* solution is

(a) $$(x+3y-z)^3+(4x-3y+2z)^3+(3x-6y+6z)^3$$
$$=(x-3y+5z)^3+(4x-6y+5z)^3.$$

(b) $$(6x+33y-24z)^3+(8x-6y+13x)^3+(19x-33y+29z)^3$$
$$+(17x+6y-8z)^3=(10x+30y-25z)^3$$
$$+(5x-15y+25z)^3+(20x-30y+25z)^3.$$

(c) $$(-4x+4y+4z)^3+(2x+3y-z)^3+(4x-2y+2z)^3$$
$$+(-2x-3y+9z)^3+(-6x+y+9z)^3$$
$$=(-6x+3y+7z)^3+(2x-3y+5z)^3$$
$$+(-2x+y+5z)^3+(-4x-2y+10z)^3.$$

Editor's Note: The statement, "if A, then B", is usually considered true in either of the three cases (i) A is never true, (ii) B is always true, (iii) B is deducible from A. In the present example, (iii) has been shown above, but both (i) and (ii) hold. It is well known that no integers all different from zero exist which satisfy $x^3+y^3=z^3$.* On the other hand, infinitely many solutions of (a), (b) and (c) actually exist. For examples: $7^3+8^3=1^3+5^3+9^3$, $7^3+9^3=2^3+4^3+10^3$ for (a)†; $4^3+7^3+11^3=1^3+2^3+9^3+10^3$ for (b); $4^3+8^3+16^3+17^3=1^3+2^3+10^3+14^3+18^3$ for (*c*) As illustrated by the last two examples, we may find infinitely many solutions for (b) and (c) even under the additional requirements that the sums of the squares and the sums of the first powers shall also be equal.‡

No. 190. Proposed by *Nathan Altshiller-Court*, University of Oklahoma.

Given the tetrahedron $ABCD$ and a point L in space. The parallel to the lines LB, LC, LD, through the mid-points of the edges AB, AC, AD, respectively, have a point P in common. We have three analogous points Q, R, S, for the vertices B, C, D. Show that the lines joining the points P, Q, R, S to the centroids of the respective faces of $ABCD$ are concurrent.

Solution by the *Proposer*.

The parallel to LB through the mid-point of the edge AB passes through the mid-point P of the segment LA. Likewise, for the parallels to LC and LD. Thus the tetrahedron determined by the point P and its analogues Q, R, S corresponds to the given tetrahedron $ABCD$ in the homothecy (L, 2) having L for homothetic center, and two for homothetic ratio. Now the tetrahedron $A'B'C'D'$ having for vertices the centroids of the faces of $ABCD$ is also homothetic to $ABCD$. Hence the two tetrahedrons $PQRS$ and $A'B'C'D'$ are homothetic,|| and the lines PA', QB', CR', DS' are concurrent.

Also solved by *W. T. Short*.

*Consult Carmichael, *Diophantine Analysis*, (1915), pp. 67-70.

†Carmichael, op. cit., p. 73, ex. 20.

‡Dorwart and Brown, *The Tarry-Escott Problem*, The American Mathematical Monthly, Vol. 44 (1937), pp. 613-636.

||Nathan Altshiller--Court, *Modern Pure Solid Geometry*, p. 18, Art. 56. (Macmillan) 1935.

No. 191. Proposed by *Nathan Altshiller-Court*, University of Oklahoma.

The tangent plane to a given sphere, center O, at a variable point M cuts a fixed plane along a line d. Find the locus of the projection, L, of the point M upon the plane passing through O and d.

Solution by *W. T. Short*, Oklahoma Baptist University.

Let the coordinate system be so chosen that

$$x^2+y^2+z^2=r^2 \tag{1}$$

is the equation of the sphere and

$$ax+by+cz=1 \tag{2}$$

the equation of the plane.

Let the coordinates of M be (x_1,y_1,z_1). The polar plane of M is

$$xx_1+yy_1+zz_1=r^2 \tag{3}$$

and this is the tangent plane if x_1,y_1,z_1 satisfy (1). The equation of the plane through the intersection of (2) and (3) and through the origin is:

$$xx_1+yy_1+zz_1=r^2\cdot(ax+by+cz). \tag{4}$$

Thus the equation of the line through M and perpendicular to plane (4) is

$$(x-x_1)/(x_1-ar^2)=(y-y_1)/(y_1-br^2)=(z-z_1)/(z_1-cr^2). \tag{5}$$

Now the coordinates of L may be found as a solution of (4) and (5). However, if we solve (4) and (5) for (x_1,y_1,z_1) and substitute in (1) it should give the locus of L. Solving these for x_1, we have:

$$(ar^2x+br^2y+cr^2z-x^2-y^2-z^2)x_1$$
$$=ar^2(ar^2x+br^2y+cr^2z-x^2-y^2-z^2).$$

Thus

$$x^2+y^2+z^2-ar^2x-br^2y-cr^2z=0 \quad \text{or} \tag{6}$$

$$x_1=ar^2, \text{ and by analogy: } y_1=br^2,\ z_1=cr^2. \tag{7}$$

But (7) is the pole of the given plane with respect to the given sphere. If point (7) were used for M, planes (2) and (3) would coincide and there would be no locus for the planes (4).

Thus we see that the sphere represented by equation (6) is the locus for all points L derived from M and that M may be any point

except the pole (2) with respect to (1). Therefore (6) is the locus if M is on the sphere. Note that the foregoing transformation transforms all points in space except one into the points on the surface of a shere.

The *Proposer* offers the following neat solution:

If the line OL meets d in the point Q, we have

$$OM^2 = OL \cdot OQ,$$

hence L and Q are inverse points with respect to the given sphere (0). Now Q is an arbitrary point of the fixed plane, for from any point Q a tangent plane may be drawn to (0), hence the locus of the point L is the sphere inverse of the given plane with respect to the given sphere (0).

If the given plane cuts the given sphere, only a part of the inverse sphere will constitute the required locus.

Note: It is regretted that a solution of No. 169 by *Walter B. Clarke*, was not acknowledged in the January issue.

PROPOSALS

No. 217. Proposed by *Walter B. Clarke*, San Jose, California.

Construct a triangle whose Euler line is parallel to a side. Under what conditions, expressed in terms of the sides, is this possible?

No. 218. Proposed by *Walter B. Clarke*, San Jose, California.

Given a circle with two intersecting chords AB and CD. Locate a point on either chord equidistant from the other chord and from the circumference.

No. 219. Proposed by *Jeannette Simpson*, student, New Jersey College for Women.

With ruler and compasses locate the point P on the side AB of any triangle ABC such that the perpendicular from P to AC is the mean proportional between AP and PB. Find also the point Q such that the line QE parallel to BC and intercepted by AC is the mean proportional between AQ and QB.

No. 220. Proposed by *G. W. Wishard*, Norwood, Ohio.

The following are easily verified: $33^2 = 2244$ in the scale of 5; $44^2 = 3344$ in the scale of 6; $55^2 = 4444$ in the scale of 7. The law of formation is readily seen; prove the series continues indefinitely.

No. 221. Proposed by *E. P. Starke*, Rutgers University.

Prove: if two roots of a cubic equation with rational coefficients have a rational difference, all the roots are rational.

No. 222. Proposed by *Alfred Moessner*, Nurnberg-N, Germany.

What is the general solution in integers of the system:

$$a+b+c+d=e+f+g+h$$

$$a^2+b^2+c^2+d^2=e^2+f^2+g^2+h^2$$

$$a^4+b^4=c^4+d^4 \quad ?$$

No. 223. Proposed by *Albert Farnell*, Centenary College, Shreveport, Louisiana.

Form numbers the sum of whose digits is 27 and which are perfect cubes in every number system in which the base is sufficiently large. Find the system of smallest base in which such a cube can be written.

No. 224. Proposed by *Dewey C. Duncan*, Compton Junior College, California.

If the external bisectors of the base angles of a scalene triangle are equal, prove that

(a) If the two sides are given, the triangle is uniquely determined, but *not* constructible by Euclidean means.

(b) If the base b and one side a $(b>a)$ are given, the triangle is uniquely determined and *is* constructible by Euclidean means.

No. 225. Proposed by *V. Thébault*, Le Mans, France.

Consider three circles (0_1), (0_2), (0_3) tangent among themselves two by two and respectively equal to three of the tritangent circles of a given triangle ABC.

(1) Find the radii of the circles (W_1) and (W_2) which are tangent to the three circles (0_1), (0_2), (0_3).

(2) Find the necessary and sufficient condition that one of the circles (W_1), (W_2) be equal to the fourth tritangent circle of the triangle.

No. 226. Proposed by *V. Thébault*, Le Mans, France.

Given any four spheres, show that the lines joining the internal centers of similitude of each pair of spheres with the other meet in a point.

Reviews and Abstracts

Edited by
P. K. SMITH and H. A. SIMMONS

The Elementary Theory of Operational Mathematics. By Eugene Stephens. McGraw-Hill, New York, 1937. xi+313 pages.

This text deals with "D" operators and their generalizations as applied to linear differential equations, ordinary and partial. The algebra of linear forms, including matrix theory, is used to classify systems of differential equations. A large number of formulas and applications drawn from many sources is presented, along with an historical account of the growth of the theory and a biographical list of papers from 1765 to date. No attempt is made to develop the theory from the more recent points of view of Laplace-transformations or of the Fourier integral theory.

There are some obvious misprints or errors on pages 16 (D is an operator, not a number), 17, 19, 36, 37, 41, 58, 64, 91, 133, 277, 286. On page 65 one might well take exception to the statement "Neither these (referring to work of Murnaghan and Carson) nor any of the many other interpretations of the theorem (Heaviside's expansion theorem) have been made clear enough for the ordinary student to use."

On page 70 there is an indictment of the teaching of algebra: "Much of the difficulty that engineering students encounter in gaining facility in the use of the operations of the calculus lies in their unfamiliarity with many of the most elegant theorems of algebra. This is partly due to the speed with which they are driven over their algebra and party due to the fact that colllege algebras do not contain some of the really useful theorems or methods. For instance, the really elegant theorems for the development of partial fractions are not in the algebra texts and in only one elementary calculus text. The most useful method of evaluating numerical determinants is not in any of the algebras in common use in America. Matrix theory is not touched upon at all in any engineering mathematics courses." It would be interesting to learn the reaction of college teachers to this paragraph.

The book contains a mine of information, but it is the opinion of the reviewer that the modern Laplace-transform point of view will eventually come to the fore even in engineering applications.

Virginia Military Institute. W. E. BYRNE.

Mechanics. By W. F. Osgood. The Macmillan Company, New York, 1937. xv+495 pages. Price $5.00.

In this new text, *Mechanics*, there are the same rigor of treatment, originality of explanation, and clarity and humor of style that characterize the other familiar texts by W. F. Osgood. Effort is made, both in the text and in the use of well selected problems, to give physical concreteness to the various concepts.

Approximately the first half of the book is intended for a first course in mechanics, and contains the usual topics such as Statics and Dynamics of Particles and Rigid Bodies. Two illustrations will suffice to show the character of Osgood's method of treatment. He has an excellent discussion of friction wherein he emphasizes the determination of all positions for equilibrium. More significantly, in dynamics he obtains a complete description of motion by integrating simple differential equations, as contrasted with the usual special case procedure.

The second half of the text is designed for an advanced course and treats such topics as Kinematics and Kinetics in Three Dimensions, Lagrange's Equations, and D'Alembert's Principle. Hamilton's Equations and Principle are carefully explained and a proof is included showing that these equations are invariant with respect to contact transformations.

An appendix contains background material on vector analysis, the differential equation: $(du/dt)^2=f(u)$, and Jacobi's Equation. There is also included a formulation of "The General Problem of Rational Mechanics" which serves to correlate the material of the text.

In the prevailing college system there are two types of courses in mechanics: one course is given in either the Physics or Mathematics Department and the other in the Engineering College. The former often over-emphasizes the theoretical point of view, whereas the latter neglects both rigor of theory and thoroughness of treatment (especially with multiple-valued functions). The first half of this text, in the reading matter and problems, contains the application point of view as well as the rigor and completeness of the theoretical course. In the second half there is naturally more emphasis on the theory. Certain of the problems throughout the book (of which some are from the Cambridge Tripos) serve to develop the student's mathematical technique.

One of the principle tasks of college teachers is to teach students how properly to read a textbook. This book should aid both teacher and student to surmount the difficulty, for its style is lucid and enjoy-

able; and its readability is heightened by the unusual footnotes, which fulfill three functions: correlation of text with prerequisite mathematics and physics, study hints to students and teacher, and historical remarks.

To the teacher of applied mechanics who may complain of the use of differential equations and calculus in the treatment of some of the topics, I should like to quote from the preface of *Communication Networks* by E. A. Guillemin: "To the student, the entire field is new; the advanced methods are no exception. If they afford a better understanding of the situation involved, then it is good pedagogy to introduce them into an elementary discussion. It is well for the teacher to bear in mind that methods which are very familiar to him are not necessarily the easiest for the student to grasp." In engineering texts there is all too often an awkward avoidance of the use of the mathematics which the student has already studied rather than the straightforward approach through this medium.

Every teacher of applied mathematics would do well to read this text both for the material and for applications of elementary college mathematics.

The typography is excellent. The reviewer has found only two errors and those are in an insufficient labeling of figures.

North Carolina State College. JOHN W. CELL.

CARL FRIEDRICH GAUSS. Inaugural Lecture on Astronomy and Papers on the Foundations of Mathematics. Translated and edited by G. Waldo Dunnington. Louisiana State University Press, Baton Rouge, 1937. xi+91 pages.

This monograph from the pen of G. Waldo Dunnington, of the mathematics faculty of the University of Illinois, will be welcomed by English-speaking admirers of the great Gauss. Especially since it deals chiefly with the early and less known stages in Gauss' career, describing with competence of knowledge and intimacy of understanding the development of Gauss, the man, and the origins of the fields of interest and choice of methods of Gauss, the mathematician. For very little of what this monograph discusses can be found in the current histories of mathematics.

Carl Friedrich Gauss, the greatest mathematician the German race has produced and one of the greatest of all times, was born of humble parentage in Brunswick, April 30, 1777. In this part of Germany he spent his entire life. He received his education at Brunswick's

Katharinen Volksschule (1784-88), the Katharineum (1788-91, the Carolineum (1792-95), and the University at Göttingen (1795-98), where he studied with equal avidity the classical languages and mathematics. After receiving the doctorate from the University of Helmstedt in 1799 he spent the next few years in research. As a result of his brilliant calculation of the elements of the planet Ceres, he was appointed director of the observatory at Göttingen, which position he held until his death, Feb. 23, 1855. It is said that after his appointment to the Göttingen post he never slept away from the observatory except on one occasion when he went to Berlin to attend a scientific congress.

From his first investigations into the theory of numbers as an 18-year old student until his death at 78 astronomy, physics, geodesy, electricity, as well as mathematics, felt the touch of his mighty hand. "This mathematical giant", to quote the elder Bolyai, "who from his lofty heights embraces the stars and the abysses", marked the turning-point in the development of three sciences: mathematics, physics, and astronomy.

"The great Gauss." But what was he before he became so great? What were his antecedents and his beginnings? What kind of a man was he? What kind of a boy was he? Dunnington does much to answer these questions for English-speaking people.

In the first part of his work Dunnington gives an interesting and detailed account of the early life of Gauss up to the year 1803. This he can do with authority; for he has studied both at Brunswick and Göttingen. He also had free access to the Gauss Archive at the Royal Society of Göttingen. We mention three exhibits the author studied. One is the little *Notizenjournal* wherein Carl Friedrich recorded his scientific discoveries from 1796 to 1801. Another is the collection of early or little known works of Gauss, two of which are given in the monograph. Another is the collection of Gauss' text books from boyhood to university days. On the fly leaves in these young Carl Friedrich would often scribble proofs and notes, and these notes are sometimes the only indications we have of the processes by which he arrived at his discoveries. For in his published works he only gives logical and synthetic expositions.

The beautiful friendship between Wolfgang Bolyai and Carl Friedrich, which began in university days and lasted through life, is well told. Wolfgang was a Magyar from Siebenbirgen. One thing that drew them together was their common respect for scientific integrity and rigor of proof. "Our moral agreement bound us together so that while out walking we were silent for hours at a time, each

occupied with his own thoughts". Came a time when Wolfgang had to go back to his distant Hungary and the friends must separate. On May 24, 1799, at sunset, they met for the last time. Says Bolyai: "He wrote that I was to determine the time and place. I decided on Clausthal and we appeared punctually, just as his stars did later. Our parting was on a mountain peak toward Brunswick". Travel conditions in Central Europe were harsh in those days, and they never saw one another again. But they kept up correspondence till old age.

The last part of the monograph consists of the two early works of Gauss mentioned in the title and the translator-editor's Introductions and Notes. These, to quote his preface, "make a systematic attempt to illuminate the text by the citation of passages which show the sources of certain ideas, various historical and etymological allusions, etc., or by references to incidents in Gauss' own life". This program is carried out commendably.

As to the two translated articles of Gauss, which occupy 38 pages, they are significant as setting forth, the one, Gauss' standards and norms for his mathematical philosophy, the other, his pedagogy in the teaching of astronomy.

The first paper bears the title *The Metaphysics of Mathematics*, but is what in modern terminology we call Foundations of Mathematics. This paper has only recently come to light. Gauss prepared it before his career had started when he was wondering whether he should teach or do research.

The second paper, his inaugural lecture as professor of astronomy at Göttingen, gives us a picture of the status of astronomy before the period of Gauss, and also acts as a standard to show the development of the science after 1807. He divides astronomical study into three stages: spherical astronomy, dealing with the apparent motions of the heavenly bodies; theoretical astronomy, ascertaining their true motions; physical astronomy, dealing with laws and causes. This reviewer notes with special interest one point in his pedagogy; that instead of giving the subject matter to his audience in synthetic form as done by most writers, he is going to give it in the form and order in which it was discovered. The author's justification for this procedure is: "First, because in the beginning steps very refined mathematical knowledge is immediately necessary; even more, however, second, because one does not see in this way by what methods one attained the knowledge and therefore does not receive a very complete and vital conviction." This method he very likely used in his class room; it is a natural method to use in lectures. But one wishes he would also have used it liberally in his published works. For Gauss' formal treatises

are notably synthetic, seldom giving any clues to his arrival at new discoveries.

The translation itself leaves something to be desired. Certain turns of phrases and the long drawn-out sentences—on page 51 one sentence comprises 17 lines—make unfamiliar and somewhat difficult reading. Could not the original sentence structure in Gauss have been broken up into smaller parts without doing violence to the scientist's meaning and rhetoric? But the reviewer can sympathize and appreciate, having himself tried his hand at translating scientific works by continental authors of this era of long sentences and dependent clauses.

Yet when that is said, Gauss' ideas stand out clear and the translator-editor's orienting comments are very helpful and illuminating. He intimates a second monograph on Gauss after 1803 may be forthcoming; we sincerely hope it will.

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